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VARIATIONS AND ESTIMATORS FOR SELF-SIMILARITY PARAMETERS VIA MALLIAVIN CALCULUS

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Using multiple stochastic integrals and the Malliavin calculus, we analyze the asymptotic behavior of quadratic variations for a specific non-Gaussian self-similar process, the Rosenblatt process. We apply our results to the design of strongly consistent statistical estimators for the self-similarity parameter H. Although, in the case of the Rosenblatt process, our estimator has non-Gaussian asymptotics for all $H > 1/2$, we show the remarkable fact that the process's data at time 1 can be used to construct a distinct, compensated estimator with Gaussian asymptotics for $H \in (1/2, 2/3)$.

1. Introduction.

1.1. *Context and motivation.* A *self-similar process* is a stochastic process such that any part of its trajectory is invariant under time scaling. Self-similar processes are of considerable interest in practice in modeling various phenomena, including internet traffic (see, e.g., [32]), hydrology (see, e.g., [13]) or economics (see, e.g., [12, 31]). In various applications, empirical data also shows strong correlation of observations, indicating the presence, in addition to self-similarity, of long-range dependence. We refer to the monographs [7] and [25] for various properties of, and fields of application for, such processes.

The motivation for this work is to examine non-Gaussian self-similar processes using tools from stochastic analysis. We will focus our attention on a special process of this type, the so-called *Rosenblatt process*. This belongs to a class of self-similar processes which also exhibit long-range dependence and

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which appear as limits in the so-called noncentral limit theorem: the class of *Hermite* processes. We study the behavior of the quadratic variations for the Rosenblatt process Z, which is related to recent results by [14, 16, 17], and we apply the results to the study of estimators for the self-similarity parameter of Z. Recently, results on variations or weighted quadratic variations of fractional Brownian motion were obtained in [14, 16, 17], among others. The Hermite processes were introduced by Taqqu (see [27] and [28]) and by Dobrushin and Major (see [5]). The Hermite process of order $q \geq 1$ can be written, for every $t \geq 0$, as

$$\begin{aligned} Z_H^q(t) &= c(H,q) \\ &\quad \times \int_{\mathbb{R}^q} \left[\int_0^t \left(\prod_{i=1}^q (s-y_i)_+^{-(1/2+(1-H)/q)} \right) ds \right] dW(y_1)\cdots dW(y_q), \end{aligned} \tag{1.1}$$

where $c(H,q)$ is an explicit positive constant depending on q and H, and such that $\mathbf{E}(Z_H^q(1)^2) = 1$, $x_+ = \max(x,0)$, the self-similarity (Hurst) parameter H belongs to the interval $(\frac{1}{2}, 1)$ and the above integral is a multiple Wiener–Itô stochastic integral with respect to a two-sided Brownian motion $(W(y))_{y\in\mathbb{R}}$ (see [21]). We note that the Hermite processes of order $q > 1$, which are non-Gaussian, have only been defined for $H > \frac{1}{2}$; how to define these processes for $H \leq \frac{1}{2}$ is still an open problem.

The case $q = 1$ is the well-known fractional Brownian motion (fBm): this is Gaussian. One recognizes that when $q = 1$, (1.1) is the moving average representation of fractional Brownian motion. The Rosenblatt process is the case $q = 2$. All Hermite processes share the following basic properties:

- they exhibit long-range dependence (the long-range covariance decays at the rate of the nonsummable power function n^{2H-2});
- they are H-self-similar, in the sense that for any $c > 0$, $(Z_H^q(ct))_{t\geq 0}$ and $(c^H Z_H^q(t))_{t\geq 0}$ are equal in distribution;
- they have stationary increments, that is, the distribution of $(Z_H^q(t+h) - Z_H^q(h))_{t\geq 0}$ does not depend on $h > 0$;
- they share the same covariance function,

  $$\mathbf{E}[Z_H^q(t) Z_H^q(s)] =: R^H(t,s) = \tfrac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}), \qquad s,t \geq 0,$$

  so, for every $s,t \geq 0$, the expected squared increment of the Hermite process is

  $$\mathbf{E}[(Z_H^q(t) - Z_H^q(s))^2] = |t-s|^{2H} \tag{1.2}$$

  from which it follows by Kolmogorov's continuity criterion, and the fact that each $L^p(\Omega)$-norm of the increment of Z_H^q over $[s,t]$ is commensurate with its $L^2(\Omega)$-norm, that this process is almost surely Hölder continuous of any order $\delta < H$;

- the qth Hermite process lives in the so-called qth Wiener chaos of the underlying Wiener process W since it is a qth order Wiener integral.

The stochastic analysis of fBm has been developed intensively in recent years and its applications are numerous. Other Hermite processes are less well studied, but are still of interest due to their long-range dependence, self-similarity and stationarity of increments. The great popularity of fBm in modeling is due to these properties and fBm is preferred over higher order Hermite processes because it is a Gaussian process and because its calculus is much easier. However, in concrete situations, when empirical data attests to the presence of self-similarity and long memory without the Gaussian property, one can use a Hermite process living in a higher chaos.

The Hurst parameter H characterizes all of the important properties of a Hermite process, as seen above. Therefore, properly estimating H is of the utmost importance. Several statistics have been introduced to this end, such as wavelets, k-variations, variograms, maximum likelihood estimators and spectral methods. Information on these various approaches can be found in the book by Beran [1].

In this paper, we will use variation statistics to estimate H. Let us recall the context. Suppose that a process $(X_t)_{t\in[0,1]}$ is observed at discrete times $\{0, \frac{1}{N}, \dots, \frac{N-1}{N}, 1\}$ and let a be a "filter" of length $l \geq 0$ and $p \geq 1$ a fixed power; that is, a is an $l+1$-dimensional vector $a = (a_0, a_1, \dots, a_l)$ such that $\sum_{q=0}^{l} a_q q^r = 0$ for $0 \leq r \leq p-1$ and $\sum_{q=0}^{l} a_q q^p \neq 0$. The k-variation statistic associated to the filter a is then defined as

$$V_N(k,a) = \frac{1}{N-l} \sum_{i=l}^{N-1} \left[\frac{|V_a(i/N)|^k}{\mathbf{E}[|V_a(i/N)|^k]} - 1 \right],$$

where, for $i \in \{l, \dots, N\}$,

$$V_a\left(\frac{i}{N}\right) = \sum_{q=0}^{l} a_q X\left(\frac{i-q}{N}\right).$$

When X is fBm, these statistics are used to derive strongly consistent estimators for the Hurst parameter and their associated normal convergence results. A detailed study can be found in [8] and [11] or, more recently, in [4]. The behavior of $V_N(k,a)$ is used to derive similar behaviors for the corresponding estimators. The basic result for fBm is that, if $p > H + \frac{1}{4}$, then the renormalized k-variation $V_N(k,a)$ converges to a standard normal distribution. The easiest and most natural case is that of the filter $a = \{1, -1\}$, in which case $p = 1$; one then has the restriction $H < \frac{3}{4}$. The techniques used to prove such convergence in the fBm case in the above references are strongly related to the Gaussian property of the observations; they appear not to extend to non-Gaussian situations.

Our purpose here is to develop new techniques that can be applied to both the fBm case and to other non-Gaussian self-similar processes. Since this is the first attempt in such a direction, we keep things as simple as possible: we treat the case of the filter $a = \{1, -1\}$ with a k-variation order $= 2$ (quadratic variation), but the method can be generalized. As announced above, we further specialize to the simplest non-Gaussian Hermite process, that is, the one of order 2, the Rosenblatt process. We now give a short overview of our results (a more detailed summary of these facts is given in the next subsection). We obtain that, after suitable normalization, the quadratic variation statistic of the Rosenblatt process converges to a Rosenblatt random variable with the same self-similarity order; in fact, this random variable is the observed value of the original Rosenblatt process at time 1 and the convergence occurs in the mean square. More precisely, the quadratic variation statistic can be decomposed into the sum of two terms: a term in the fourth Wiener chaos (i.e., an iterated integral of order 4 with respect to the Wiener process) and a term in the second Wiener chaos. The fourth Wiener chaos term is well behaved, in the sense that it has a Gaussian limit in distribution, but the second Wiener chaos term is ill behaved, in the sense that its asymptotics are non-Gaussian and are, in fact, Rosenblatt-distributed. This term, being of a higher order than the well-behaved one, is responsible for the asymptotics of the entire statistic. But, since its convergence occurs in the mean-square and the limit is observed, we can construct an adjusted variation by subtracting the contribution of the ill-behaved term. We find an estimator for the self-similarity parameter of the Rosenblatt process, based on observed data, whose asymptotic distribution is normal.

Our main tools are the Malliavin calculus, the Wiener–Itô chaos expansions and recent results on the convergence of multiple stochastic integrals proved in [10, 22, 23] and [24]. The key point is the following: if the observed process X lives in some Wiener chaos of finite order, then the statistic V_N can be decomposed, using product formulas and Wiener chaos calculus, into a finite sum of multiple integrals. One can then attempt to apply the criteria in [22] to study the convergence in law of such sequences and to derive asymptotic normality results (or to demonstrate the lack thereof) on the estimators for the Hurst parameter of the observed process. The criteria in [22] are necessary and sufficient conditions for convergence to the Gaussian law; in some instances, these criteria fail (e.g., the fBm case with $H > 3/4$), in which case, a proof of nonnormal convergence "by hand," working directly with the chaoses, can be employed. It is the basic Wiener chaos calculus that makes this possible.

1.2. *Summary of results.* We now summarize the main results of this paper in some detail. As stated above, we use quadratic variation with $a = \{1, -1\}$. We consider the two following processes, observed at the discrete

times $\{i/N\}_{i=0}^{N}$: the fBm process $X = B$ and the Rosenblatt process $X = Z$. In either case, the standardized quadratic variation and the Hurst parameter estimator are given, respectively, by

$$V_N = V_N(2, \{-1, 1\}) := \frac{1}{N} \sum_{i=1}^{N} \left(\frac{|X(i/N) - X((i-1)/N)|^2}{N^{-2H}} - 1 \right), \tag{1.3}$$

$$\hat{H}_N = \hat{H}_N(2, \{-1, 1\}) := \frac{1}{2} - \frac{1}{2 \log N} \log \sum_{i=1}^{N} \left(X\left(\frac{i}{N}\right) - X\left(\frac{i-1}{N}\right) \right)^2. \tag{1.4}$$

We choose to use the normalization $\frac{1}{N}$ in the definition of V_N (as, e.g., in [4]), although, in the literature, it sometimes does not appear. The H-dependent constants $c_{j,H}$ (et al.) referred to below are defined explicitly in (3.2), (3.6), (3.12), (3.14), (3.21) and (3.33). Here, and throughout, $L^2(\Omega)$ denotes the set of square-integrable random variables measurable with respect to the sigma field generated by W. This sigma-field is the same as that generated by B or by Z. The term "Rosenblatt random variable" denotes a random variable whose distribution is the same as that of $Z(1)$.

We first recall the followings facts, relative to fractional Brownian motion:

1. if $X = B$ and $H \in (1/2, 3/4)$, then:
   (a) $\sqrt{N/c_{1,H}}\, V_N$ converges in distribution to the standard normal law;
   (b) $\sqrt{N} \log(N) \frac{2}{\sqrt{c_{1,H}}} (\hat{H}_N - H)$ converges in distribution to the standard normal law;
2. if $X = B$ and $H \in (3/4, 1)$, then:
   (a) $\sqrt{N^{4-4H}/c_{2,H}}\, V_N$ converges in $L^2(\Omega)$ to a standard Rosenblatt random variable with parameter $H_0 = 2H - 1$;
   (b) $N^{1-H} \log(N) \frac{2}{\sqrt{c_{2,H}}} (\hat{H}_N - H)$ converges in $L^2(\Omega)$ to the same standard Rosenblatt random variable;
3. if $X = B$ and $H = 3/4$, then:
   (a) $\sqrt{N/(c'_{1,H} \log N)}\, V_N$ converges in distribution to the standard normal law;
   (b) $\sqrt{N \log N} \frac{2}{\sqrt{c'_{1,H}}} (\hat{H}_N(2, a) - H)$ converges in distribution to the standard normal law.

The convergences for the standardized V_N's in points 1(a) and 2(a) have been known for some time, in works such as [28] and [9]. Lately, even stronger results, which also give error bounds, have been proven. We refer to [19] for the one-dimensional case and $H \in (0, \frac{3}{4})$, [2] for then one-dimensional case and $H \in [\frac{3}{4}, 1)$ and to [20] for the multidimensional case and $H \in (0, \frac{3}{4})$.

In this paper, we prove the following results for the Rosenblatt process $X = Z$ as $N \to \infty$:

4. if $X = Z$ and $H \in (1/2, 1)$, then with $c_{3,H}$ in (3.12),
   (a) $N^{1-H} V_N(2, a)/(c_{3,H})$ converges in $L^2(\Omega)$ to the Rosenblatt random variable $Z(1)$;
   (b) $\frac{N^{1-H}}{2c_{3,H}} \log(N)(\hat{H}_N(2, a) - H)$ converges in $L^2(\Omega)$ to the same Rosenblatt random variable $Z(1)$;
5. if $X = Z$ and $H \in (1/2, 2/3)$, then, with $e_{1,H}$ and $f_{1,H}$ in (3.21) and (3.33),
   (a) $\frac{\sqrt{N}}{\sqrt{e_{1,H}+f_{1,H}}}[V_N(2, a) - \frac{\sqrt{c_{3,H}}}{N^{1-H}} Z(1)]$ converges in distribution to the standard normal law;
   (b) $\frac{\sqrt{N}}{\sqrt{e_{1,H}+f_{1,H}}}[2\log(N)(H - \hat{H}_N(2, a)) - \frac{\sqrt{c_{3,H}}}{N^{1-H}} Z(1)]$ converges in distribution to the standard normal law.

Note that $Z(1)$ is the *actual observed value* of the Rosenblatt process at time 1, which is why it is legitimate to include it in a formula for an estimator. Points 4 and 5 are new results. The subject of variations and statistics for the Rosenblatt process has thus far received too narrow a treatment in the literature, presumably because standard techniques inherited from the noncentral limit theorem (and sometimes based on the Fourier transform formula for the driving Gaussian process) are difficult to apply (see [3, 5, 28]). Our Wiener chaos calculus approach allows us to show that the standardized quadratic variation and corresponding estimator both converge to a Rosenblatt random variable in $L^2(\Omega)$. Here, our method has a crucial advantage: we are able to determine which Rosenblatt random variable it converges to: it is none other than the observed value $Z(1)$. The fact that we are able to prove $L^2(\Omega)$ convergence, not just convergence in distribution, is crucial. Indeed, when $H < 2/3$, subtracting an appropriately normalized version of this observed value from the quadratic variation and its associated estimator, we prove that asymptotic normality does hold in this case. This unexpected result has important consequences for the statistics of the Rosenblatt process since it permits the use of standard techniques in parameter estimation and testing.

Our asymptotic normality result for the Rosenblatt process was specifically made possible by showing that V_N can be decomposed into two terms: a term T_4 in the fourth Wiener chaos and a term T_2 in the second Wiener chaos. While the second-Wiener-chaos term T_2 always converges to the Rosenblatt random variable $Z(1)$, the fourth chaos term T_4 converges to a Gaussian random variable for $H \leq 3/4$. We conjecture that this asymptotic normality should also occur for Hermite processes of higher order $q \geq 3$ and that the threshold $H = 3/4$ is universal. The threshold $H < 2/3$ in the results above comes from the discrepancy that exists between a normalized T_2 and its observed limit $Z(1)$. If we were to rephrase results 4 and 5 above, with T_2 instead of $Z(1)$ (which is not a legitimate operation when defining

an estimator since T_2 is not observed), the threshold would be $H \le 3/4$ and the constant $f_{1,H}$ would vanish.

Beyond our basic interest concerning parameter estimation problems, let us situate our paper in the context of some recent and interesting works on the asymptotic behavior of p-variations (or weighted variations) for Gaussian processes, namely the papers [14, 15, 16, 17] and [26]. These recent papers study the behavior of sequences of the type

$$\sum_{i=1}^{N} h(X((i-1)/N))\left(\frac{|X(i/N)-X((i-1)/N)|^2}{N^{-2H}}-1\right),$$

where X is a Gaussian process (fractional Brownian motion in [14, 16] and [17], and the solution of the heat equation driven by a space-time white noise in [26]) or the iterated Brownian motion in [18] and h is a regular deterministic function. In the fractional Brownian motion case, the behavior of such sums varies according to the values of the Hurst parameter, the limit sometimes being a conditionally Gaussian random variable, sometimes a deterministic Riemann integral and sometimes a pathwise integral with respect to a Hermite process. We believe that our work is the first to tackle a non-Gaussian case, that is, when the process X above is a Rosenblatt process. Although we restrict ourselves to the case when $h \equiv 1$, we still observe the appearance of interesting limits, depending on the Hurst parameter: while, in general, the limit of the suitably normalized sequence is a Rosenblatt random variable (with the same Hurst parameter H as the data, which poses a slight problem for statistical applications), the adjusted variations (i.e., the sequences obtained by subtracting precisely the portion responsible for the non-Gaussian convergence) do converge to a Gaussian limit for $H \in (1/2, 2/3)$.

This article is structured as follows. Section 2 presents preliminaries on fractional stochastic analysis. Section 3 contains proofs of our results for the non-Gaussian Rosenblatt process. Some calculations are recorded as lemmas that are proven in the Appendix. Section 4 establishes our parameter estimation results, which follow almost trivially from the theorems in Section 3.

2. Preliminaries. Here, we describe the elements from stochastic analysis that we will need in the paper. Consider $\mathcal{H}$, a real, separable Hilbert space and $(B(\varphi), \varphi \in \mathcal{H})$, an isonormal Gaussian process, that is, a centered Gaussian family of random variables such that $\mathbf{E}(B(\varphi)B(\psi)) = \langle \varphi, \psi \rangle_{\mathcal{H}}$.

Denote by I_n the multiple stochastic integral with respect to B (see [21] and [30]). This I_n is actually an isometry between the Hilbert space $\mathcal{H}^{\odot n}$ (symmetric tensor product) equipped with the scaled norm $\frac{1}{\sqrt{n!}}\|\cdot\|_{\mathcal{H}^{\otimes n}}$ and the Wiener chaos of order n which is defined as the closed linear span of the

random variables $H_n(B(\varphi))$, where $\varphi \in \mathcal{H}$, $\|\varphi\|_{\mathcal{H}} = 1$ and H_n is the Hermite polynomial of degree n.

We recall that any square-integrable random variable which is measurable with respect to the σ-algebra generated by B can be expanded into an orthogonal sum of multiple stochastic integrals,

$$F = \sum_{n \geq 0} I_n(f_n),$$

where $f_n \in \mathcal{H}^{\odot n}$ are (uniquely determined) symmetric functions and $I_0(f_0) = \mathbf{E}[F]$.

In this paper, we actually use only multiple integrals with respect to the standard Wiener process with time horizon $[0,1]$ and, in this case, we will always have $\mathcal{H} = L^2([0,1])$. This notation will be used throughout the paper.

We will need the general formula for calculating products of Wiener chaos integrals of any orders, p and q, for any symmetric integrands $f \in \mathcal{H}^{\odot p}$ and $g \in \mathcal{H}^{\odot q}$; it is

$$I_p(f)I_q(g) = \sum_{r=0}^{p \wedge q} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}(f \otimes_r g), \tag{2.1}$$

as given, for instance, in Nualart's book [21], Proposition 1.1.3; the contraction $f \otimes_r g$ is the element of $\mathcal{H}^{\otimes(p+q-2r)}$ defined by

$$\begin{aligned}(f \otimes_r g)&(s_1, \ldots, s_{p-r}, t_1, \ldots, t_{q-r}) \\ &= \int_{[0,T]^{p+q-2r}} f(s_1, \ldots, s_{p-r}, u_1, \ldots, u_r) \\ &\qquad \times g(t_1, \ldots, t_{q-r}, u_1, \ldots, u_r)\, du_1 \cdots du_r.\end{aligned} \tag{2.2}$$

We now introduce the Malliavin derivative for random variables in a chaos of finite order. If $f \in \mathcal{H}^{\odot n}$, we will use the following rule to differentiate in the Malliavin sense:

$$D_t I_n(f) = n I_{n-1}(f_n(\cdot, t)), \qquad t \in [0,1].$$

It is possible to characterize the convergence in distribution of a sequence of multiple integrals to the standard normal law. We will use the following result (see Theorem 4 in [22], also [23]).

THEOREM 2.1. *Fix $n \geq 2$ and let $(F_k, k \geq 1)$, $F_k = I_n(f_k)$ (with $f_k \in \mathcal{H}^{\odot n}$ for every $k \geq 1$), be a sequence of square-integrable random variables in the nth Wiener chaos such that $\mathbf{E}[F_k^2] \to 1$ as $k \to \infty$. The following are then equivalent:*

(i) *the sequence $(F_k)_{k \geq 0}$ converges in distribution to the normal law $\mathcal{N}(0,1)$;*

(ii) $\mathbf{E}[F_k^4] \to 3$ *as $k \to \infty$;*

(iii) *for all* $1 \le l \le n-1$, *it holds that* $\lim_{k\to\infty} \|f_k \otimes_l f_k\|_{\mathcal{H}^{\otimes 2(n-l)}} = 0$;
(iv) $\|DF_k\|^2_{\mathcal{H}} \to n$ *in* $L^2(\Omega)$ *as* $k \to \infty$, *where* D *is the Malliavin derivative with respect to* B.

Criterion (iv) is due to [22]; we will refer to it as the *Nualart–Ortiz-Latorre criterion*. A multidimensional version of the above theorem has been proven in [24] (see also [22]).

3. Variations for the Rosenblatt process. Our observed process is a Rosenblatt process $(Z(t))_{t\in[0,1]}$ with self-similarity parameter $H \in (\frac{1}{2}, 1)$. This centered process is self-similar with stationary increments and lives in the second Wiener chaos. Its covariance is identical to that of fractional Brownian motion. Our goal is to estimate its self-similarity parameter H from discrete observations of its sample paths. As far as we know, this direction has seen little or no attention in the literature and the classical techniques (e.g., the ones from [5, 27] and [28]) do not work well for it. Therefore, the use of the Malliavin calculus and multiple stochastic integrals is of interest.

The Rosenblatt process can be represented as follows (see [29]): for every $t \in [0,1]$,

$$\begin{aligned} Z^H(t) &:= Z(t) \\ &= d(H) \int_0^t \int_0^t \left[\int_{y_1\vee y_2}^t \partial_1 K^{H'}(u,y_1) \right. \\ &\qquad \left. \times\, \partial_1 K^{H'}(u,y_2)\, du \right] dW(y_1)\, dW(y_2), \end{aligned} \tag{3.1}$$

where $(W(t), t\in[0,1])$ is some standard Brownian motion, $K^{H'}$ is the standard kernel of fractional Brownian motion of index H' (see any reference on fBm, such as [21], Chapter 5) and

$$H' = \frac{H+1}{2} \quad \text{and} \quad d(H) = \frac{(2(2H-1))^{1/2}}{(H+1)H^{1/2}}. \tag{3.2}$$

For every $t \in [0,1]$, we will denote the kernel of the Rosenblatt process with respect to W by

$$\begin{aligned} L_t^H(y_1,y_2) &:= L_t(y_1,y_2) \\ &:= d(H)\left[\int_{y_1\vee y_2}^t \partial_1 K^{H'}(u,y_1)\,\partial_1 K^{H'}(u,y_2)\,du\right] 1_{[0,t]^2}(y_1,y_2). \end{aligned} \tag{3.3}$$

In other words, in particular, for every t,

$$Z(t) = I_2(L_t(\cdot)),$$

where I_2 denotes the multiple integral of order 2 introduced in Section 2.

Now, consider the filter $a = \{-1, 1\}$ and the 2-variations given by

$$\begin{aligned} V_N(2,a) &= \frac{1}{N}\sum_{i=1}^{N} \frac{(Z(i/N) - Z((i-1)/N))^2}{\mathbf{E}(Z(i/N) - Z((i-1)/N))^2} - 1 \\ &= N^{2H-1}\sum_{i=1}^{N}\left[\left(Z\left(\frac{i}{N}\right) - Z\left(\frac{i-1}{N}\right)\right)^2 - N^{-2H}\right]. \end{aligned}$$

The product formula for multiple Wiener–Itô integrals (2.1) yields

$$I_2(f)^2 = I_4(f \otimes f) + 4I_2(f \otimes_1 f) + 2\|f\|^2_{L^2([0,1]^2)}.$$

Setting, for $i = 1, \dots, N$,

$$A_i := L_{i/N} - L_{(i-1)/N}, \tag{3.4}$$

we can thus write

$$\left(Z\left(\frac{i}{N}\right) - Z\left(\frac{i-1}{N}\right)\right)^2 = (I_2(A_i))^2 = I_4(A_i \otimes A_i) + 4I_2(A_i \otimes_1 A_i) + N^{-2H}$$

and this implies that the 2-variation is decomposed into a fourth chaos term and a second chaos term:

$$V_N(2,a) = N^{2H-1}\sum_{i=1}^{N}(I_4(A_i \otimes A_i) + 4I_2(A_i \otimes_1 A_i)) := T_4 + T_2.$$

A detailed study of the two terms above will shed light on some interesting facts: if $H \le \frac{3}{4}$, then the term T_4 continues to exhibit "normal" behavior (when renormalized, it converges in law to a Gaussian distribution), while the term T_2, which turns out to be dominant, never converges to a Gaussian law. One can say that the second Wiener chaos portion is "ill behaved"; however, once it is subtracted, one obtains a sequence converging to $\mathcal{N}(0,1)$ for $H \in (\frac{1}{2}, \frac{2}{3})$, which has an impact on statistical applications.

3.1. *Expectation evaluations.*

3.1.1. *The term T_2.* Let us evaluate the mean square of the second term,

$$T_2 := 4N^{2H-1}\sum_{i=1}^{N} I_2(A_i \otimes_1 A_i).$$

We use the notation $I_i = (\frac{i-1}{N}, \frac{i}{N}]$ for $i = 1, \dots, N$. The contraction $A_i \otimes_1 A_i$ is given by

$$\begin{aligned} &(A_i \otimes_1 A_i)(y_1, y_2) \\ &\quad = \int_0^1 A_i(x, y_1)A_i(x, y_2)\,dx \end{aligned}$$

$$
\begin{aligned}
&= d(H)^2 \int_0^1 dx\, 1_{[0,i/N]}(y_1 \vee x) 1_{[0,i/N]}(y_2 \vee x) \\
&\qquad \times \biggl(\int_{x \vee y_1}^{i/N} \partial_1 K^{H'}(u,x)\, \partial_1 K^{H'}(u,y_1)\, du \\
&\qquad\quad - 1_{[0,(i-1)/N]}(y_1 \vee x) \\
&\qquad\qquad \times \int_{x \vee y_1}^{(i-1)/N} \partial_1 K^{H'}(u,x)\, \partial_1 K^{H'}(u,y_1)\, du \biggr) \\
&\qquad \times \biggl(\int_{x \vee y_2}^{i/N} \partial_1 K^{H'}(v,x)\, \partial_1 K^{H'}(v,y_2)\, dv \\
&\qquad\quad - 1_{[0,(i-1)/N]}(y_2 \vee x) \\
&\qquad\qquad \times \int_{x \vee y_2}^{(i-1)/N} \partial_1 K^{H'}(v,x)\, \partial_1 K^{H'}(v,y_2)\, dv \biggr).
\end{aligned} \tag{3.5}
$$

Defining

$$
a(H) := H'(2H'-1) = H(H+1)/2, \tag{3.6}
$$

note the following fact (see [21], Chapter 5):

$$
\int_0^{u \wedge v} \partial_1 K^{H'}(u,y_1)\, \partial_1 K^{H'}(v,y_1)\, dy_1 = a(H)|u-v|^{2H'-2}; \tag{3.7}
$$

in fact, this relation can easily be derived from $\int_0^{u\wedge v} K^{H'}(u,y_1) K^{H'}(v,y_1)\, dy_1 = R^{H'}(u,v)$ and will be used repeatedly in the sequel.

To use this relation, we first expand the product in the expression for the contraction in (3.5), taking care to keep track of the indicator functions. The resulting initial expression for $(A_i \otimes_1 A_i)(y_1,y_2)$ contains four terms, which are all of the following form:

$$
\begin{aligned}
C_{a,b} := d(H)^2 \int_0^1 dx\, 1_{[0,a]}(y_1 \vee x) 1_{[0,b]}(y_2 \vee x) \\
\times \int_{u = y_1 \vee x}^{a} \partial_1 K^{H'}(u,x)\, \partial_1 K^{H'}(u,y_1)\, du \\
\times \int_{v = y_2 \vee x}^{b} \partial_1 K^{H'}(v,x)\, \partial_1 K^{H'}(v,y_2)\, dv.
\end{aligned}
$$

Here, to perform a Fubini argument by bringing the integral over x inside, we first note that $x < u \wedge v$ while $u \in [y_1, a]$ and $v \in [y_2, b]$. Also, note that the conditions $x \le u$ and $u \le a$ imply that $x \le a$ and thus $1_{[0,a]}(y_1 \vee x)$ can be replaced, after Fubini, by $1_{[0,a]}(y_1)$. Therefore, using (3.7), the above

expression equals

$$\begin{aligned}
C_{a,b} &= d(H)^2 1_{[0,a]\times[0,b]}(y_1,y_2) \int_{y_1}^{a} \partial_1 K^{H'}(u,y_1)\,du \int_{y_2}^{b} \partial_1 K^{H'}(v,y_2)\,dv \\
&\qquad\qquad \times \int_0^{u\wedge v} \partial_1 K^{H'}(u,x)\,\partial_1 K^{H'}(v,x)\,dx \\
&= d(H)^2 a(H) 1_{[0,a]\times[0,b]}(y_1,y_2) \int_{u=y_1}^{a}\int_{v=y_2}^{b} \partial_1 K^{H'}(u,y_1)\,\partial_1 K^{H'}(v,y_2) \\
&\qquad\qquad \times |u-v|^{2H'-2}\,du\,dv \\
&= d(H)^2 a(H) \int_{u=y_1}^{a}\int_{v=y_2}^{b} \partial_1 K(u,y_1)\,\partial_1 K^{H'}(v,y_2)|u-v|^{2H'-2}\,du\,dv.
\end{aligned}$$

The last equality comes from the fact that the indicator functions in y_1, y_2 are redundant: they can be pulled back into the integral over $du\,dv$ and, therein, the functions $\partial_1 K^{H'}(u,y_1)$ and $\partial_1 K^{H'}(v,y_2)$ are, by definition as functions of y_1 and y_2, supported by smaller intervals than $[0,a]$ and $[0,b]$, namely $[0,u]$ and $[0,v]$, respectively.

Now, the contraction $(A_i \otimes_1 A_i)(y_1,y_2)$ equals $C_{i/N,i/N} + C_{(i-1)/N,(i-1)/N} - C_{(i-1)/N,i/N} - C_{i/N,(i-1)/N}$. Therefore, from the last expression above,

$$\begin{aligned}
&(A_i \otimes_1 A_i)(y_1,y_2) \\
&\quad= a(H)d(H)^2 \Bigg(\int_{y_1}^{i/N} du \int_{y_2}^{i/N} dv\,\partial_1 K^{H'}(u,y_1) \\
&\qquad\qquad \times \partial_1 K^{H'}(v,y_2)|u-v|^{2H'-2} \\
&\qquad\qquad - \int_{y_1}^{i/N} du \int_{y_2}^{(i-1)/N} dv\,\partial_1 K^{H'}(u,y_1) \\
&\qquad\qquad \times \partial_1 K^{H'}(v,y_2)|u-v|^{2H'-2} \\
&\qquad\qquad - \int_{y_1}^{(i-1)/N} du \int_{y_2}^{i/N} dv\,\partial_1 K^{H'}(u,y_1) \\
&\qquad\qquad \times \partial_1 K^{H'}(v,y_2)|u-v|^{2H'-2} \\
&\qquad\qquad + \int_{y_1}^{(i-1)/N} du \int_{y_2}^{(i-1)/N} dv\,\partial_1 K^{H'}(u,y_1) \\
&\qquad\qquad \times \partial_1 K^{H'}(v,y_2)|u-v|^{2H'-2} \Bigg).
\end{aligned} \tag{3.8}$$

Since the integrands in the above four integrals are identical, we can simplify the above formula, grouping the first two terms, for instance, to obtain an integral of v over $I_i = (\frac{i-1}{N}, \frac{i}{N}]$, with integration over u in $[y_1, \frac{i}{n}]$. The same

operation on the last two terms gives the negative of the same integral over v, with integration over u in $[y_1, \frac{i-1}{n}]$. Then, grouping these two resulting terms yields a single term, which is an integral for (u,v) over $I_i \times I_i$. We obtain the following, final, expression for our contraction:

$$\begin{aligned}(A_i \otimes_1 A_i)(y_1,y_2) \\ = a(H)d(H)^2 \iint_{I_i\times I_i} \partial_1 K^{H'}(u,y_1)\,\partial_1 K^{H'}(v,y_2)|u-v|^{2H'-2}\,du\,dv.\end{aligned} \tag{3.9}$$

Now, since the integrands in the double Wiener integrals defining T_2 are symmetric, we get

$$\mathbf{E}[T_2^2] = N^{4H-2} 16\cdot 2! \sum_{i,j=1}^{N} \langle A_i \otimes_1 A_i, A_j \otimes_1 A_j\rangle_{L^2([0,1]^2)}.$$

To evaluate the inner product of the two contractions, we first use Fubini with expression (3.9); by doing so, one must realize that the support of $\partial_1 K^{H'}(u,y_1)$ is $\{u > y_1\}$, which then makes the upper limit 1 for the integration in y_1 redundant; similar remarks hold with respect to u', v, v' and y_2. In other words, we have

$$\begin{aligned}
&\langle A_i \otimes_1 A_i, A_j \otimes_1 A_j\rangle_{L^2([0,1])^2} \\
&\quad = a(H)^2 d(H)^4 \int_0^1\int_0^1 dy_1\,dy_2 \int_{I_i}\int_{I_i}\int_{I_j}\int_{I_j} du'\,dv'\,du\,dv \\
&\qquad \times |u-v|^{2H'-2}|u'-v'|^{2H'-2}\,\partial_1 K^{H'}(u,y_1)\,\partial_1 K^{H'}(v,y_2) \\
&\qquad \times \partial_1 K^{H'}(u',y_1)\,\partial_1 K^{H'}(v',y_2) \\
&\quad = a(H)^2 d(H)^4 \int_{I_i}\int_{I_i}\int_{I_j}\int_{I_j} |u-v|^{2H'-2}|u'-v'|^{2H'-2}\,du'\,dv'\,dv\,du \qquad (3.10) \\
&\qquad\qquad \times \int_0^{u\wedge u'} \partial_1 K^{H'}(u,y_1)\,\partial_1 K^{H'}(u',y_1)\,dy_1 \\
&\qquad\qquad \times \int_0^{v\wedge v'} \partial_1 K^{H'}(v,y_2)\,\partial_1 K^{H'}(v',y_2)\,dy_2 \\
&\quad = a(H)^4 d(H)^4 \int_{I_i}\int_{I_i}\int_{I_j}\int_{I_j} |u-v|^{2H'-2}|u'-v'|^{2H'-2}|u-u'|^{2H'-2} \\
&\qquad\qquad\qquad \times |v-v'|^{2H'-2}\,du'\,dv'\,dv\,du,
\end{aligned}$$

where we have used the expression (3.7) in the last step. Therefore, we immediately have

$$\mathbf{E}[T_2^2] = N^{4H-2} 32 a(H)^4 d(H)^4$$

$$\times \sum_{i,j=1}^{N} \int_{I_i}\int_{I_i}\int_{I_j}\int_{I_j} du'\,dv'\,dv\,du \tag{3.11}$$
$$\times |u-v|^{2H'-2}|u'-v'|^{2H'-2}|u-u'|^{2H'-2}|v-v'|^{2H'-2}.$$

By Lemma 5 in the Appendix, we conclude that

$$\begin{aligned}\lim_{N\to\infty} \mathbf{E}[T_2^2]N^{2-2H} &= 64a(H)^2 d(H)^4\left(\frac{1}{2H-1}-\frac{1}{2H}\right)\\ &= 16d(H)^2 \\ &:= c_{3,H}.\end{aligned}\tag{3.12}$$

3.1.2. *The term T_4.* Now, for the L^2-norm of the term denoted by

$$T_4 := N^{2H-1}\sum_{i=1}^{N} I_4(A_i\otimes A_i),$$

by the isometry formula for multiple stochastic integrals, and using a correction term to account for the fact that the integrand in T_4 is nonsymmetric, we have

$$\begin{aligned}\mathbf{E}[T_4^2] &= 8N^{4H-2}\sum_{i,j=1}^{N}\langle A_i\otimes A_i; A_j\otimes A_j\rangle_{L^2([0,1]^4)}\\ &\quad + 4N^{4H-2}\sum_{i,j=1}^{N} 4\langle A_i\otimes_1 A_j; A_j\otimes_1 A_i\rangle_{L^2([0,1]^2)} =: \mathcal{T}_{4,0}+\mathcal{T}_{4,1}.\end{aligned}$$

We separate the calculation of the two terms $\mathcal{T}_{4,0}$ and $\mathcal{T}_{4,1}$ above. We will see that these two terms are exactly of the same magnitude, so both calculations must be performed precisely.

The first term, $\mathcal{T}_{4,0}$, can be written as

$$\mathcal{T}_{4,0} = 8N^{4H-2}\sum_{i,j=1}^{N} |\langle A_i, A_j\rangle_{L^2([0,1]^2)}|^2.$$

We calculate each individual scalar product $\langle A_i, A_j\rangle_{L^2([0,1]^2)}$ as

$$\begin{aligned}&\langle A_i, A_j\rangle_{L^2([0,1]^2)}\\ &\quad = \int_0^1\int_0^1 A_i(y_1,y_2)A_j(y_1,y_2)\,dy_1\,dy_2\\ &\quad = d(H)^2\int_0^1\int_0^1 dy_1\,dy_2\,1_{[0,i/N\wedge j/N]}(y_1\vee y_2)\end{aligned}$$

$$\times \left(\int_{y_1 \vee y_2}^{i/N} \partial_1 K^{H'}(u, y_1)\, \partial_1 K^{H'}(u, y_2)\, du - 1_{[0,(i-1)/N]}(y_1 \vee y_2) \right.$$
$$\left. \times \int_{y_1 \vee y_2}^{(i-1)/N} \partial_1 K^{H'}(u, y_1)\, \partial_1 K^{H'}(u, y_2)\, du \right)$$
$$\times \left(\int_{y_1 \vee y_2}^{j/N} \partial_1 K^{H'}(v, y_1) \partial_1 K^{H'}(v, y_2)\, dv - 1_{[0,(j-1)/N]}(y_1 \vee y_2) \right.$$
$$\left. \times \int_{y_1 \vee y_2}^{(j-1)/N} \partial_1 K^{H'}(v, y_1)\, \partial_1 K^{H'}(v, y_2)\, dv \right)$$
$$= d(H)^2 \int_{(i-1)/N}^{i/N} \int_{(j-1)/N}^{j/N} du\, dv \left[\int_0^{u \wedge v} \partial_1 K^{H'}(u, y_1)\, \partial_1 K^{H'}(v, y_1)\, dy_1 \right]^2.$$

Here, (3.7) yields

$$\langle A_i, A_j \rangle_{L^2([0,1]^2)} = d(H)^2 a(H)^2 \int_{I_i} \int_{I_j} |u - v|^{2H-2}\, du\, dv,$$

where, we have again used the notation $I_i = (\frac{i-1}{N}, \frac{i}{N}]$ for $i = 1, \ldots, N$. We finally obtain

$$\langle A_i, A_j \rangle_{L^2([0,1]^2)} = \frac{d(H)^2 a(H)^2}{H(2H-1)} \frac{1}{2} \left[2\left| \frac{i-j}{N} \right|^{2H} - \left| \frac{i-j+1}{N} \right|^{2H} - \left| \frac{i-j-1}{N} \right|^{2H} \right], \tag{3.13}$$

where, more precisely, $d(H)^2 a(H)^2 (H(2H-1))^{-1} = 2$. Specifically, with the constants $c_{1,H}$, $c_{2,H}$ and $c'_{1,H}$ given by

$$\begin{aligned} c_{1,H} &:= 2 + \sum_{k=1}^{\infty} (2k^{2H} - (k-1)^{2H} - (k+1)^{2H})^2, \\ c_{2,H} &:= 2H^2(2H-1)/(4H-3), \\ c'_{1,H} &:= (2H(2H-1))^2 = 9/16, \end{aligned} \tag{3.14}$$

using Lemmas 3, 4 and an analogous result for $H = 3/4$, we get, asymptotically for large N,

$$\lim_{N \to \infty} N \mathcal{T}_{4,0} = 16 c_{1,H}, \qquad 1/2 < H < \tfrac{3}{4}, \tag{3.15}$$

$$\lim_{N \to \infty} N^{4-4H} \mathcal{T}_{4,0} = 16 c_{2,H}, \qquad H > \tfrac{3}{4}, \tag{3.16}$$

$$\lim_{N \to \infty} \frac{N}{\log N} \mathcal{T}_{4,0} = 16 c'_{1,H} = 16, \qquad H = \frac{3}{4}. \tag{3.17}$$

The second term, $\mathcal{T}_{4,1}$, can be dealt with by obtaining an expression for

$$\langle A_i \otimes_1 A_j; A_j \otimes_1 A_i \rangle_{L^2([0,1]^2)}$$

in the same way as the expression obtained in (3.10). We get

$$\begin{aligned}\mathcal{T}_{4,1} &= 16N^{4H-2}\sum_{i,j=1}^{N}\langle A_i \otimes_1 A_j; A_j \otimes_1 A_i\rangle_{L^2([0,1]^2)}\\ &= 16d(H)^4 a(H)^4 N^{-2}\sum_{i,j=1}^{N}\int_0^1\int_0^1\int_0^1\int_0^1 dy\,dz\,dy'\,dz'\\ &\qquad\times|y-z+i-j|^{2H'-2}|y'-z'+i-j|^{2H'-2}\\ &\qquad\times|y-y'+i-j|^{2H'-2}|z-z'+i-j|^{2H'-2}.\end{aligned}$$

Now, similarly to the proof of Lemma 5, we find the the following three asymptotic behaviors:

- if $H\in(\frac{1}{2},\frac{3}{4})$, then $\tau_{1,H}^{-1}N\mathcal{T}_{4,1}$ converges to 1, where

$$\tau_{1,H} := 16d(H)^4 a(H)^4 c_{1,H}; \tag{3.18}$$

- if $H>\frac{3}{4}$, then $\tau_{2,H}^{-1}N^{4-4H}\mathcal{T}_{4,1}$ converges to 1, where

$$\tau_{2,H} := 32d(H)^4 a(H)^4\int_0^1(1-x)x^{4H-4}\,dx; \tag{3.19}$$

- if $H=\frac{3}{4}$, then $\tau_{3,H}^{-1}(N/\log N)\mathcal{T}_{4,1}$ converges to 1, where

$$\tau_{3,H} := 32d(H)^4 a(H)^4. \tag{3.20}$$

Combining these results for $\mathcal{T}_{4,1}$ with those for $\mathcal{T}_{4,0}$ in lines (3.15), (3.16) and (3.17), we obtain the asymptotics of $\mathbf{E}[T_4^2]$ as $N\to\infty$:

$$\begin{aligned}\lim_{N\to\infty} N\mathbf{E}[T_4^2] &= e_{1,H}, &&\text{if } H\in(\tfrac{1}{2},\tfrac{3}{4});\\ \lim_{N\to\infty} N^{4-4H}\mathbf{E}[T_4^2] &= e_{2,H}, &&\text{if } H\in(\tfrac{3}{4},1);\\ \lim_{N\to\infty}\frac{N}{\log N}\mathbf{E}[T_4^2] &= e_{3,H}, &&\text{if } H=\frac{3}{4},\end{aligned}$$

where, with $\tau_{i,H}$, $i=1,2,3$, given in (3.18), (3.19) and (3.20), we defined

$$\begin{aligned}e_{1,H} &:= (1/2)c_{1,H}+\tau_{1,H},\\ e_{2,H} &:= (1/2)c_{2,H}+\tau_{2,H},\\ e_{3,H} &:= c_{3,H}+\tau_{3,H}.\end{aligned} \tag{3.21}$$

Taking into account the estimations (3.15), (3.16) and (3.17), with $c_{3,H}$ in (3.12), we see that $\mathbf{E}[T_4^2]$ is always of smaller order than $\mathbf{E}[T_2^2]$; therefore,

the mean-square behavior of V_N is given by that of the term T_2 only, which means that we obtain, for every $H > 1/2$,

$$\lim_{N\to\infty} \mathbf{E}\left[\left(N^{1-H} V_N(2,a)\frac{1}{\sqrt{c_{3,H}}}\right)^2\right] = 1. \tag{3.22}$$

3.2. *Normality of the fourth chaos term T_4 when $H \le 3/4$.* The calculations for T_4 above prove that $\lim_{N\to\infty} \mathbf{E}[G_N^2] = 1$ for $H < 3/4$, where $e_{1,H}$ is given in (3.21) and

$$G_N := \sqrt{N} N^{2H-1} e_{1,H}^{-1/2} I_4\left(\sum_{i=1}^N A_i \otimes A_i\right). \tag{3.23}$$

Similarly, for $H = \frac{3}{4}$, we showed that $\lim_{N\to\infty} \mathbf{E}[\tilde{G}_N^2] = 1$, where $e_{3,H}$ is given in (3.21) and

$$\tilde{G}_N := \sqrt{\frac{N}{\log N}} N^{2H-1} e_{3,H}^{-1} I_4\left(\sum_{i=1}^N A_i \otimes A_i\right). \tag{3.24}$$

Using the criterion of Nualart and Ortiz-Latorre [part (iv) in Theorem 2.1], we prove the following asymptotic normality for G_N and $\tilde{G}_N$.

THEOREM 3.1. *If $H \in (1/2, 3/4)$, then G_N given by (3.23) converges in distribution as*

$$\lim_{N\to\infty} G_N = \mathcal{N}(0,1). \tag{3.25}$$

If $H = 3/4$, then $\tilde{G}_N$ given by (3.24) converges in distribution as

$$\lim_{N\to\infty} \tilde{G}_N = \mathcal{N}(0,1). \tag{3.26}$$

PROOF. We will denote by c a generic positive constant not depending on N.

STEP 0 (Setup and expectation evaluation). Using the derivation rule for multiple stochastic integrals, the Malliavin derivative of G_N is

$$D_r G_N = \sqrt{N} N^{2H-1} e_{1,H}^{-1/2} 4 \sum_{i=1}^N I_3((A_i \otimes A_i)(\cdot, r))$$

and its norm is

$$\|DG_N\|_{L^2([0,1])}^2 = N^{4H-1} 16 e_{1,H}^{-1} \sum_{i,j=1}^N \int_0^1 dr\, I_3((A_i \otimes A_i)(\cdot, r)) I_3((A_j \otimes A_j)(\cdot, r)).$$

The product formula (2.1) gives

$$
\begin{aligned}
&\|DG_N\|^2_{L^2([0,1])}\\
&\quad = N^{4H-1} 16 e^{-1}_{1,H}\\
&\qquad \times \sum_{i,j=1}^{N} \int_0^1 dr\, [I_6((A_i \otimes A_i)(\cdot, r) \otimes (A_j \otimes A_j)(\cdot, r))\\
&\qquad\qquad + 9 I_4((A_i \otimes A_i)(\cdot, r) \otimes_1 (A_j \otimes A_j)(\cdot, r))\\
&\qquad\qquad + 9 I_2((A_i \otimes A_i)(\cdot, r) \otimes_2 (A_j \otimes A_j)(\cdot, r))\\
&\qquad\qquad + 3! I_0((A_i \otimes A_i)(\cdot, r) \otimes_3 (A_j \otimes A_j)(\cdot, r))]\\
&\quad =: J_6 + J_4 + J_2 + J_0.
\end{aligned}
$$

First, note that, for the nonrandom term J_0 that gives the expected value of the above, we have

$$
\begin{aligned}
J_0 &= 16 e^{-1}_{1,H} N^{4H-1} 3! \sum_{i,j=1}^{N} \int_{[0,1]^4} A_i(y_1, y_2) A_i(y_3, y_4) A_j(y_1, y_2)\\
&\qquad \times A_j(y_3, y_4)\, dy_1\, dy_2\, dy_3\, dy_4\\
&= 96 N^{4H-1} e^{-1}_{1,H} \sum_{i,j=1}^{N} |\langle A_i, A_j \rangle_{L^2([0,1]^2)}|^2.
\end{aligned}
$$

This sum has already been treated: we know from (3.15) that $J_0/4$ converges to 1, that is, that $\lim_{N\to\infty} \mathbf{E}[\|DG_N\|^2_{L^2([0,1])}] = 4$. This means, by the Nualart–Ortiz-Latorre criterion, that we only need to show that all other terms J_6, J_4, J_2 converge to zero in $L^2(\Omega)$ as $N \to \infty$.

STEP 1 (Order-6 chaos term). We first consider the term J_6:

$$
\begin{aligned}
J_6 &= c N^{4H-1} \sum_{i,j=1}^{N} \int_0^1 dr\, I_6((A_i \otimes A_i)(\cdot, r) \otimes (A_j \otimes A_j(\cdot, r)))\\
&= c N^{4H-1} \sum_{i,j=1}^{N} I_6((A_i \otimes A_j) \otimes (A_i \otimes_1 A_j)).
\end{aligned}
$$

We study the mean square of this term. We have, since the L^2-norm of the symmetrization is less than the L^2-norm of the corresponding unsymmetrized function,

$$
\mathbf{E}\left[\left(\sum_{i,j=1}^{N} I_6((A_i \otimes A_j) \otimes (A_i \otimes_1 A_j))\right)^2\right]
$$

$$\leq 6! \sum_{i,j,k,l} \langle (A_i \otimes A_j) \otimes (A_i \otimes_1 A_j), (A_k \otimes A_l) \otimes (A_k \otimes_1 A_l) \rangle_{L^2([0,1]^6)}$$

$$= 6! \sum_{i,j,k,l} \langle A_i, A_k \rangle_{L^2([0,1]^2)} \langle A_j, A_l \rangle_{L^2([0,1]^2)} \langle A_i \otimes_1 A_j, A_k \otimes_1 A_l \rangle_{L^2([0,1]^2)}.$$

We get

$$\begin{aligned}\mathbf{E}[J_6^2] \leq cN^{8H-2} \sum_{i,j,k,l} \int_{I_i} du \int_{I_j} dv \int_{I_k} du' \int_{I_l} dv' \\ \times |u-v|^{2H'-2} |u-u'|^{2H'-2} |v-v'|^{2H'-2} |u'-v'|^{2H'-2} \\ \times \left[2\left|\frac{i-k}{N}\right|^{2H} - \left|\frac{i-k+1}{N}\right|^{2H} - \left|\frac{i-k-1}{N}\right|^{2H} \right] \\ \times \left[2\left|\frac{j-l}{N}\right|^{2H} - \left|\frac{j-l+1}{N}\right|^{2H} - \left|\frac{j-l-1}{N}\right|^{2H} \right].\end{aligned}$$

First, we show that for $H \in (1/2, 3/4)$, we have, for large N,

$$\mathbf{E}[J_6^2] \leq cN^{8H-6}. \tag{3.27}$$

With the notation as in Step 1 of this proof, making the change of variables $\bar{u} = (u - \frac{i-1}{N})N$, and similarly for the other integrands, we obtain

$$\begin{aligned}\mathbf{E}[J_6^2] &\leq cN^{8H-2} \frac{1}{N^{8H'-8}} \frac{1}{N^4} \frac{1}{N^{4H}} \\ &\quad \times \sum_{i,j,k,l} \int_{[0,1]^4} du\, dv\, du'\, dv' \\ &\quad \times |u-v+i-j|^{2H'-2} |u-u'+i-k|^{2H'-2} \\ &\quad \times |u'-v+j-k|^{2H'-2} |v-v'+k-l|^{2H'-2} \\ &\quad \times (2|i-k|^{2H} - |i-k+1|^{2H} - |i-k-1|^{2H}) \\ &\quad \times (2|j-l|^{2H} - |j-l+1|^{2H} - |j-l-1|^{2H}) \\ &= c\frac{1}{N^2} \sum_{i,j,k,l} \int_{[0,1]^4} du\, dv\, du'\, dv' \\ &\quad \times |u-v+i-j|^{2H'-2} |u-u'+i-k|^{2H'-2} \\ &\quad \times |u'-v+j-k|^{2H'-2} |v-v'+k-l|^{2H'-2} \\ &\quad \times (2|i-k|^{2H} - |i-k+1|^{2H} - |i-k-1|^{2H}) \\ &\quad \times (2|j-l|^{2H} - |j-l+1|^{2H} - |j-l-1|^{2H}).\end{aligned}$$

Again, we use the fact that the dominant part in the above expression is the one in where all indices are distant by at least two units. In this case,

up to a constant, we have the upper bound $|i-k|^{2H-2}$ for the quantity $(2|i-k|^{2H}-|i-k+1|^{2H}-|i-k-1|^{2H})$. By using Riemann sums, we can write

$$\mathbf{E}[J_6^2] \le c\frac{1}{N^2}N^4\left(\frac{1}{N^4}\sum_{i,j,k,l} f\left(\frac{i}{N},\frac{j}{N},\frac{k}{N},\frac{l}{N}\right)\right)N^{8H'-8}N^{4H-4},$$

where f is a Riemann integrable function on $[0,1]^4$ and the Riemann sum converges to the finite integral of f therein. Estimate (3.27) follows.

STEP 2 (Chaos terms of orders 4 and 2). To treat the term

$$J_4 = cN^{4H-1}\sum_{i,j=1}^{N}\int_0^1 dr\, I_4((A_i\otimes A_i)(\cdot,r)\otimes_1 (A_j\otimes A_j)(\cdot,r)),$$

since $I_4(g)=I_4(\tilde{g})$, where $\tilde{g}$ denotes the symmetrization of the function g, we can write

$$\begin{aligned} J_4 &= cN^{4H-1}\sum_{i,j=1}^{N}\langle A_i,A_j\rangle_{L^2(0,1]^2} I_4(A_i\otimes A_j)\\ &\quad + cN^{4H-1} I_4 \sum_{i,j=1}^{N}(A_i\otimes_1 A_j)\otimes(A_i\otimes_1 A_j)\\ &=: J_{4,1}+J_{4,2}. \end{aligned}$$

Both terms above have been treated in previous computations. To illustrate it, the first summand $J_{4,1}$ can be bounded above as follows:

$$\begin{aligned} \mathbf{E}|J_{4,1}|^2 &\le cN^{8H-2}\sum_{i,j,k,l=1}^{N}\langle A_i,A_j\rangle_{L^2([0,1]^2)}\langle A_i,A_k\rangle_{L^2([0,1]^2)}\\ &\qquad\qquad \times \langle A_k,A_l\rangle_{L^2([0,1]^2)}\langle A_j,A_l\rangle_{L^2([0,1]^2)}\\ &= cN^{8H-2}\sum_{i,j,k,l=1}^{N}\left[\left(\frac{i-j+1}{N}\right)^{2H}+\left(\frac{i-j-1}{N}\right)^{2H}-2\left(\frac{i-j}{N}\right)^{2H}\right]\\ &\qquad\qquad \times\left[\left(\frac{i-k+1}{N}\right)^{2H}\right.\\ &\qquad\qquad\quad \left.+\left(\frac{i-k-1}{N}\right)^{2H}-2\left(\frac{i-k}{N}\right)^{2H}\right]\\ &\qquad\qquad \times\left[\left(\frac{j-l+1}{N}\right)^{2H}\right. \end{aligned}$$

$$+\left(\frac{j-l-1}{N}\right)^{2H}-2\left(\frac{j-l}{N}\right)^{2H}\Big]$$
$$\times\left[\left(\frac{k-l+1}{N}\right)^{2H}+\left(\frac{k-l-1}{N}\right)^{2H}-2\left(\frac{k-l}{N}\right)^{2H}\right]$$

and, using the same bound $c|i-j|^{2H-2}$ for the quantity $|i-j+1|^{2H}+|i-j-1|^{2H}-2|i-j|^{2H}$ when $|i-j|\geq 2$, we obtain

$$\begin{aligned}\mathbf{E}|J_{4,1}|^2 &\leq cN^{8H-2}N^{-8H}\sum_{i,j,k,l=1}^{N}|i-j|^{2H-2}|i-k|^{2H-2}|j-l|^{2H-2}|k-l|^{2H-2}\\ &\leq cN^{8H-6}\frac{1}{N^4}\sum_{i,j,k,l=1}^{2}\frac{|i-j|^{2H-2}|i-k|^{2H-2}|j-l|^{2H-2}|k-l|^{2H-2}}{N^{4(2H-2)}}.\end{aligned}$$

This tends to zero at the speed N^{8H-6} as $N\to\infty$ by a Riemann sum argument since $H<\frac{3}{4}$.

One can also show that $\mathbf{E}|J_{4,2}|^2$ converges to zero at the same speed because

$$\begin{aligned}\mathbf{E}|J_{4,2}|^2 &= cN^{8H-2}\sum_{i,j,k,l=1}^{N}\langle(A_i\otimes_1 A_j),(A_k\otimes_1 A_l)\rangle^2_{L^2([0,1]^2)}\\ &\leq N^{8H-2}N^{-2(8H'-8)}N^{-8}\\ &\quad\times\sum_{i,j,k,l=1}^{N}\bigg(\int_{[0,1]^4}(|u-v+i-j|\\ &\qquad\qquad\times|u'-v'+k-l|\\ &\qquad\qquad\times|u-u'+i-k|\\ &\qquad\qquad\times|v-v'+j-l|)^{2H'-2}\,dv'\,du'\,dv\,du\bigg)^2\\ &\leq cN^{8H-6}.\end{aligned}$$

Thus, we obtain

$$\mathbf{E}[J_4^2]\leq cN^{8H-6}. \tag{3.28}$$

A similar behavior can be obtained for the last term J_2 by repeating the above arguments:

$$\mathbf{E}[J_2^2]\leq cN^{8H-6}. \tag{3.29}$$

STEP 3 (Conclusion). Combining (3.27), (3.28) and (3.29) and recalling the convergence result for $\mathbf{E}[T_4^2]$ proven in the previous subsection, we can

apply the Nualart–Ortiz-Latorre criterion and use the same method as in the case $H < \frac{3}{4}$ for $H = 3/4$, to conclude the proof. $\square$

3.3. *Nonnormality of the second chaos term T_2 and limit of the 2-variation.* This paragraph studies the asymptotic behavior of the term denoted by T_2 which appears in the decomposition of $V_N(2,a)$. Recall that this is the dominant term, given by

$$T_2 = 4N^{2H-1} I_2\left(\sum_{i=1}^{N} A_i \otimes_1 A_i\right),$$

and, with $\sqrt{c_{3,H}} = 4d(H)$ given in (3.12), we have shown that

$$\lim_{N\to\infty} \mathbf{E}[(N^{1-H} T_2 c_{3,H}^{-1/2})^2] = 1.$$

With $T_N := N^{1-H} T_2 c_{3,H}^{-1/2}$, one can show that in $L^2(\Omega)$,

$$\lim_{N\to\infty} \|DT_N\|^2_{L^2([0,1])} = 2 + c,$$

where c is a strictly positive constant. As a consequence, the Nualart–Ortiz-Latorre criterion can be used to deduce that the T_N do not converge to the standard normal law. However, it is straightforward to find the limit of T_2, and thus of V_N, in $L^2(\Omega)$, in this case. We have the following result.

THEOREM 3.2. *For all $H \in (1/2, 1)$, the normalized 2-variation $N^{1-H} V_N(2, a)/(4d(H))$ converges in $L^2(\Omega)$ to the Rosenblatt random variable $Z(1)$. Note that this is the actual observed value of the Rosenblatt process at time 1.*

PROOF. Since we already proven that $N^{1-H} T_4$ converges to 0 in $L^2(\Omega)$, it is sufficient to prove that $N^{1-H} T_2/(4d(H)) - Z(1)$ converges to 0 in $L^2(\Omega)$. Since T_2 is a second-chaos random variable, that is, is of the form $I_2(f_N)$, where f_N is a symmetric function in $L^2([0,1]^2)$, it is sufficient to prove that

$$\frac{N^{1-H}}{4d(H)} f_N$$

converges to L_1 in $L^2([0,1]^2)$, where L_1 is given by (3.3). From (3.9), we get

$$\begin{aligned} f_N(y_1, y_2) &= 4N^{2H-1} a(H) d(H)^2 \\ &\quad \times \sum_{i=1}^{N} \biggl(\iint_{I_i \times I_i} |u - v|^{2H'-2} \\ &\qquad \times \partial_1 K^{H'}(u, y_1)\, \partial_1 K^{H'}(v, y_2)\, du\, dv \biggr). \end{aligned} \tag{3.30}$$

We now show that $\frac{N^{1-H}}{4d(H)} f_N$ converges pointwise, for $y_1, y_2 \in [0,1]$, to the kernel of the Rosenblatt random variable. On the interval $I_i \times I_i$, we may replace the evaluation of $\partial_1 K^{H'}$ and $\partial_1 K^{H'}$ at u and v by setting $u = v = i/N$. We then get that $f_N(y_1, y_2)$ is asymptotically equivalent to

$$\begin{aligned}
&4N^{2H-1} a(H) d(H)^2 \sum_{i=1}^{N} \mathbf{1}_{i/N \geq y_1 \vee y_2}\, \partial_1 K^{H'}(i/N, y_1)\, \partial_1 K^{H'}(i/N, y_2) \\
&\qquad\qquad \times \iint_{I_i \times I_i} du\, dv\, |u-v|^{2H'-2} \\
&\quad = 4N^{H-1} d(H)^2 \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}_{i/N \geq y_1 \vee y_2}\, \partial_1 K^{H'}(i/N, y_1)\, \partial_1 K^{H'}(i/N, y_2),
\end{aligned}$$

where we have used the identity $\iint_{I_i \times I_i} du\, dv\, |u-v|^{2H'-2} = a(H)^{-1} N^{-2H'} = a(H)^{-1} N^{-H-1}$. Therefore, we can write, for every $y_1, y_2 \in (0,1)^2$, by invoking a Riemann sum approximation,

$$\begin{aligned}
&\lim_{N\to\infty} \frac{N^{1-H}}{4d(H)} f_N(y_1, y_2) \\
&\quad = d(H) \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}_{i/N \geq y_1 \vee y_2}\, \partial_1 K^{H'}(i/N, y_1)\, \partial_1 K^{H'}(i/N, y_2) \\
&\quad = d(H) \int_{y_1 \vee y_2}^{1} \partial_1 K^{H'}(u, y_1) \partial_1 K^{H'}(u, y_2)\, du = L_1(y_1, y_2).
\end{aligned}$$

To complete the proof, it suffices to check that the sequence $(4d(H))^{-1} \times N^{1-H} f_N$ is Cauchy in $L^2([0,1]^2)$ [indeed, this implies that $(4d(H))^{-1} N^{1-H} f_N$ has a limit in $L^2([0,1]^2)$, which obviously coincides with the a.e. limit L_1 and then the multiple integral $I_2((4d(H))^{-1} N^{1-H} f_N)$ will converge to $I_2(L_1)$]. This can be checked by means of a straightforward calculation. Indeed, one has, with $C(H)$ a positive constant not depending on M and N,

$$\begin{aligned}
&\|N^{1-H} f_N - M^{1-H} f_M\|^2_{L^2([0,1]^2)} \\
&\quad = C(H) N^{2H} \sum_{i,j=1}^{N} \int_{I_i} \int_{I_i} \int_{I_j} \int_{I_j} |u-v|^{2H'-2} |u'-v'|^{2H'-2} \\
&\qquad\qquad\qquad \times |u-u'|^{2H'-2} |v-v'|^{2H'-2}\, du'\, dv'\, du\, dv \\
&\qquad + C(H) M^{2H} \\
&\qquad \times \sum_{i,j=1}^{M} \int_{(i-1)/M}^{i/M} \int_{(i-1)/M}^{i/M} \int_{(j-1)/M}^{j/M} \int_{(j-1)/M}^{j/M} |u-v|^{2H'-2} |u'-v'|^{2H'-2}
\end{aligned}$$

$$\times |u-u'|^{2H'-2} \times |v-v'|^{2H'-2}\,du'\,dv'\,du\,dv \tag{3.31}$$

$$- 2C(H)M^{1-H}N^{1-H}M^{2H-1}N^{2H-1} \times \sum_{i=1}^{N}\sum_{j=1}^{M}\int_{I_i}\int_{I_i}\int_{(j-1)/M}^{j/M}\int_{(j-1)/M}^{j/M} du'\,dv'\,du\,dv \times |u-v|^{2H'-2}|u'-v'|^{2H'-2} \times |u-u'|^{2H'-2}|v-v'|^{2H'-2}.$$

The first two terms have already been studied in Lemma 5. We have shown that

$$N^{2H}\sum_{i,j=1}^{N}\int_{I_i}\int_{I_i}\int_{I_j}\int_{I_j}|u-v|^{2H'-2}|u'-v'|^{2H'-2} \times |u-u'|^{2H'-2}|v-v'|^{2H'-2}\,du'\,dv'\,du\,dv$$

converges to $(a(H)^2H(2H-1))^{-1}$. Thus, each of the first two terms in (3.31) converge to $C(H)$ times that same constant as M,N go to infinity. By the change of variables which has already been used several times, $\bar{u}=(u-\frac{i}{N})N$, the last term in (3.31) is equal to

$$\begin{aligned}
&C(H)(MN)^H\frac{1}{N^2M^2}(NM)^{2H'-2} \\
&\quad\times\sum_{i=1}^{N}\sum_{j=1}^{M}\int_{[0,1]^4}du\,dv\,du'\,dv' \\
&\qquad\times|u-v|^{2H'-2}|u'-v'|^{2H'-2} \\
&\qquad\times\left|\frac{u}{N}-\frac{u'}{M}+\frac{i}{N}-\frac{j}{M}\right|^{2H'-2} \\
&\qquad\times\left|\frac{v}{N}-\frac{v'}{M}+\frac{i}{N}-\frac{j}{M}\right|^{2H'-2} \\
&=\frac{C(H)}{MN}\sum_{i=1}^{N}\sum_{j=1}^{M}\int_{[0,1]^4}du\,dv\,du'\,dv' \\
&\qquad\times|u-v|^{2H'-2}|u'-v'|^{2H'-2} \\
&\qquad\times\left|\frac{u}{N}-\frac{u'}{M}+\frac{i}{N}-\frac{j}{M}\right|^{2H'-2}
\end{aligned}$$

$$\times \left|\frac{v}{N} - \frac{v'}{M} + \frac{i}{N} - \frac{j}{M}\right|^{2H'-2}.$$

For large i, j, the term $\frac{u}{N} - \frac{u'}{M}$ in front of $\frac{i}{N} - \frac{j}{M}$ is negligible and can be ignored. Therefore, the last term in (3.31) is equivalent to a Riemann sum than tends, as $M, N \to \infty$, to the constant $(\int_0^1 \int_0^1 |u-v|^{2H'-2}\,du\,dv)^2 \int_0^1 \int_0^1 |x-y|^{2(2H'-2)}$. This is precisely equal to $2(a(H)^2 H(2H-1))^{-1}$, that is, the limit of the sum of the first two terms in (3.31). Since the last term has a leading negative sign, the announced Cauchy convergence is established, completing the proof of the theorem. $\square$

REMARK 1. One can show that the 2-variations $V_N(2,a)$ converge to zero almost surely as N goes to infinity. Indeed, the results in this section already show that $V_N(2,a)$ converges to 0 in $L^2(\Omega)$, and thus in probability, as $N \to \infty$; the almost sure convergence is obtained by using an argument in [4] (proof of Proposition 1) based on Theorem 6.2 in [6] which gives the equivalence between the almost sure convergence and the mean-square convergence for empirical means of discrete stationary processes. This almost-sure convergence can also be proven by hand in the following standard way. Since $V_N(2,a)$ is in the fourth Wiener chaos, it is known that its $2p$th moment is bounded above by $c_p(\mathbf{E}[(V_N(2,a))^2])^{p/2}$, where c_p depends only on p. By choosing p large enough, via Chebyshev's inequality, the Borel–Cantelli lemma yields the desired conclusion.

3.4. *Normality of the adjusted variations.* According to Theorem 3.2, which we just proved, in the Rosenblatt case, the standardization of the random variable $V_N(2,a)$ does not converge to the normal law. But, this statistic, which can be written as $V_N = T_4 + T_2$, has a small *normal part*, which is given by the asymptotics of the term T_4, as we can see from Theorem 3.1. Therefore, $V_N - T_2$ will converge (under suitable scaling) to the Gaussian distribution. Of course, the term T_2, which is an iterated stochastic integral, is not practical because it cannot be observed. But, replacing it with its limit $Z(1)$ (this is observed), one can define an adjusted version of the statistic V_N that converges, after standardization, to the standard normal law.

The proof of this fact is somewhat delicate. If we are to subtract a multiple of $Z(1)$ from V_N in order to recuperate T_4 and hope for a normal convergence, the first calculation would have to be as follows:

$$\begin{aligned} V_N(2,a) - \frac{\sqrt{c_{3,H}}}{N^{1-H}} Z(1) &= V_N(2,a) - T_2 + T_2 - \frac{\sqrt{c_{3,H}}}{N^{1-H}} Z(1) \\ &= T_4 + \frac{\sqrt{c_{3,H}}}{N^{1-H}} \left[\frac{N^{1-H}}{\sqrt{c_{3,H}}} T_2 - Z(1)\right] \qquad (3.32) \\ &:= T_4 + U_2. \end{aligned}$$

The term T_4, when normalized as $\frac{\sqrt{N}}{\sqrt{e_{1,H}}}T_4$, converges to the standard normal law, by Theorem 3.1. To get a normal convergence for the entire expression in (3.32), one may hope that the additional term $U_2 := \frac{\sqrt{c_{3,H}}}{N^{1-H}}[\frac{N^{1-H}}{\sqrt{c_{3,H}}}T_2 - Z(1)]$ goes to 0 "fast enough." It is certainly true that U_2 does go to 0, as we just saw in Theorem 3.2. However, the proof of that theorem did not investigate the speed of this convergence of U_2. For this convergence to be "fast enough," one must multiply the expression by the rate $\sqrt{N}$ which is needed to ensure the normal convergence of T_4: we would need $U_2 \ll N^{-1/2}$. Unfortunately, this is not true. A more detailed calculation will show that U_2 is precisely of order $\sqrt{N}$. This means that we should investigate whether $\sqrt{N}U_2$ itself converges in distribution to a normal law. Unexpectedly, this turns out to be true if (and only if) $H < 2/3$.

PROPOSITION 2. *With U_2 as defined in (3.32) and $H < 2/3$, we have that $\sqrt{N}U_2$ converges in distribution to a centered normal with variance equal to*

$$f_{1,H} := 32d(H)^4 a(H)^2 \sum_{k=1}^{\infty} k^{2H-2} F\left(\frac{1}{k}\right), \tag{3.33}$$

where the function F is defined by

$$\begin{aligned} F(x) = \int_{[0,1]^4} & du\,dv\,du'\,dv' |(u-u')x+1|^{2H'-2} \\ & \times [a(H)^2(|u-v||u'-v'||(v-v')x+1|)^{2H'-2} \\ & \quad - 2a(H)(|u-v||(v-u')x+1|)^{2H'-2} \\ & \qquad + |(u-u')x+1|^{2H'-2}]. \end{aligned} \tag{3.34}$$

Before proving this proposition, let us take note of its consequence.

THEOREM 3.3. *Let $(Z(t), t \in [0,1])$ be a Rosenblatt process with self-similarity parameter $H \in (1/2, 2/3)$ and let previous notation for constants prevail. Then, the following convergence occurs in distribution:*

$$\lim_{N\to\infty} \frac{\sqrt{N}}{\sqrt{e_{1,H} + f_{1,H}}}\left[V_N(2,a) - \frac{\sqrt{c_{3,H}}}{N^{1-H}}Z(1)\right] = \mathcal{N}(0,1).$$

PROOF. By the considerations preceding the statement of Proposition 2, and (3.32) in particular, we have that

$$\sqrt{N}\left[V_N(2,a) - \frac{\sqrt{c_{3,H}}}{N^{1-H}}Z(1)\right] = \sqrt{N}T_4 + \sqrt{N}U_2.$$

Theorem 3.1 proves that $\sqrt{N}T_4$ converges in distribution to a centered normal with variance $e_{1,H}$. Proposition 2 proves that $\sqrt{N}U_2$ converges in distribution to a centered normal with variance $f_{1,H}$. Since these two sequences of random variables live in two distinct chaoses (fourth and second respectively), Theorem 1 in [24] implies that the sum of these two sequences converges in distribution to a centered normal with variance $e_{1,H} + f_{1,H}$. The theorem is proved. $\square$

To prove Proposition 2, we must first perform the calculation which yields the constant $f_{1,H}$ therein. This result is postponed to the Appendix, as Lemma 6; it shows that $\mathbf{E}[(\sqrt{N}U_2)^2]$ converges to $f_{1,H}$. Another (very) technical result needed for the proof of Proposition 2, which is used to guarantee that $\sqrt{N}U_2$ has a normal limiting distribution, is also included in the Appendix as Lemma 7. An explanation of why the conclusions of Proposition 2 and Theorem 3.3 cannot hold when $H \geq 2/3$ is also given in the Appendix, after the proof of Lemma 7. We now prove the proposition.

Proof of Proposition 2. Since U_2 is a member of the second chaos, we introduce notation for its kernel. We write

$$\frac{\sqrt{N}}{\sqrt{f_{1,H}}}U_2 = I_2(g_N),$$

where g_N is the following symmetric function in $L^2([0,1]^2)$:

$$g_N(y_1, y_2) := \frac{N^{H-1/2}}{\sqrt{f_{1,H}}}\left(\frac{N^{1-H}}{4d(H)} f_N(y_1, y_2) - L_1(y_1, y_2)\right).$$

Lemma 6 proves that $\mathbf{E}[(I_2(g_N))^2] = \|g_N\|^2_{L^2([0,1]^2)}$ converges to 1 as $N \to \infty$. By the result in [23] for second-chaos sequences (see Theorem 1, point (ii) in [23], which is included as part (iii) of Theorem 2.1 herein), we have that $I_2(g_N)$ will converge to a standard normal if (and only if)

$$\lim_{N\to\infty} \|g_N \otimes_1 g_N\|^2_{L^2([0,1]^2)} = 0,$$

which would complete the proof of the proposition. This fact does hold if $H < 2/3$. We have included this technical and delicate calculation as Lemma 7 in the Appendix. Following the proof of this lemma is a discussion of why the above limit cannot be 0 when $H \geq 2/3$. $\square$

4. The estimators for the self-similarity parameter. In this section, we construct estimators for the self-similarity exponent of a Hermite process based on the discrete observations of the driving process at times $0, \frac{1}{N}, \dots, 1$. It is known that the asymptotic behavior of the statistics $V_N(2, a)$ is related

to the asymptotic properties of a class of estimators for the Hurst parameter H. This is mentioned in, for instance, [4].

We recall the setup for how this works. Suppose that the observed process X is a Hermite process; it may be Gaussian (fractional Brownian motion) or non-Gaussian (Rosenblatt process, or even a higher order Hermite process). With $a = \{-1, +1\}$, the 2-variation is denoted by

$$S_N(2,a) = \frac{1}{N}\sum_{i=1}^{N}\left(X\left(\frac{i}{N}\right) - X\left(\frac{i-1}{N}\right)\right)^2. \tag{4.1}$$

Recall that $\mathbf{E}[S_N(2,a)] = N^{-2H}$. By estimating $\mathbf{E}[S_N(2,a)]$ by $S_N(2,a)$, we can construct the estimator

$$\hat{H}_N(2,a) = -\frac{\log S_N(2,a)}{2\log N}, \tag{4.2}$$

which coincides with the definition in (1.4) given at the beginning of this paper. To prove that this is a strongly consistent estimator for H, we begin by writing

$$1 + V_N(2,a) = S_N(2,a)N^{2H},$$

where V_N is the original quantity defined in (1.3), and thus

$$\log(1 + V_N(2,a)) = \log S_N(2,a) + 2H\log N = -2(\hat{H}_N(2,a) - H)\log N.$$

Moreover, by Remark 1, $V_N(2,a)$ converges almost surely to 0 and thus $\log(1 + V_N(2,a)) = V_N(2,a)(1 + o(1))$, where $o(1)$ converges to 0 almost surely as $N \to \infty$. Hence, we obtain

$$V_N(2,a) = 2(H - \hat{H}_N(2,a))(\log N)(1 + o(1)). \tag{4.3}$$

Relation (4.3) means that the V_N's behavior immediately gives the behavior of $\hat{H}_N - H$.

Specifically, we can now state our convergence results. In the Rosenblatt data case, the renormalized error $\hat{H}_N - H$ does not converge to the normal law. But, from Theorem 3.3, we can obtain an adjusted version of this error that converges to the normal distribution.

THEOREM 4.1. *Suppose that $H > \frac{1}{2}$ and that the observed process Z is a Rosenblatt process with self-similarity parameter H. Then, strong consistency holds for $\hat{H}_N$, that is, almost surely,*

$$\lim_{N\to\infty} \hat{H}_N(2,a) = H. \tag{4.4}$$

In addition, we have the following convergence in $L^2(\Omega)$:

$$\lim_{N\to\infty} \frac{N^{1-H}}{2d(H)}\log(N)(\hat{H}_N(2,a) - H) = Z(1), \tag{4.5}$$

where $Z(1)$ is the observed process at time 1.

Moreover, if $H < 2/3$, then, in distribution as $N \to \infty$, with $c_{3,H}$, $e_{1,H}$ and $f_{1,H}$ in (3.12), (3.21) and (3.33),

$$\frac{\sqrt{N}}{\sqrt{e_{1,H} + f_{1,H}}}\left[-2\log(N)(\hat{H}_N(2,a) - H) - \frac{\sqrt{c_{3,H}}}{N^{1-H}} Z(1)\right] \to \mathcal{N}(0,1).$$

PROOF. This follows from Theorems 3.3 and 3.2 and relation (4.3). □

APPENDIX

LEMMA 3. *The series $\sum_{k=1}^{\infty}(2k^{2H} - (k-1)^{2H} - (k+1)^{2H})^2$ is finite if and only if $H \in (1/2, 3/4)$.*

PROOF. Since $2k^{2H} - (k-1)^{2H} - (k+1)^{2H} = k^{2H} f(\frac{1}{k})$, with $f(x) := 2 - (1-x)^{2H} - (1+x)^{2H}$ being asymptotically equivalent to $2H(2H-1)x^2$ for small x, the general term of the series is equivalent to $(2H)^2(2H-1)^2 k^{4H-4}$. □

LEMMA 4. *When $H \in (3/4, 1)$, $N^2 \sum_{i,j=1,\dots,N; |i-j|\ge 2}(2|\frac{i-j}{N}|^{2H} - |\frac{i-j-1}{N}|^{2H} - |\frac{i-j+1}{N}|^{2H})^2$ converges to $H^2(2H-1)/(H-3/4)$ as $N \to \infty$.*

PROOF. This is left to the reader. The proof can be found in the extended version of this paper, available at `http://arxiv.org/abs/0709.3896v2`. □

LEMMA 5. *For all $H > 1/2$, with $I_i = (\frac{i-1}{N}, \frac{i}{N}]$, $i = 1, \dots, N$,*

$$\begin{aligned} \lim_{N\to\infty} N^{2H} \sum_{i,j=1}^{N} \int_{I_i}\int_{I_i}\int_{I_j}\int_{I_j} & |u-v|^{2H'-2} \\ & \times |u'-v'|^{2H'-2}|u-u'|^{2H'-2} \\ & \times |v-v'|^{2H'-2}\, du'\, dv'\, dv\, du \\ = 2a(H)^{-2}\left(\frac{1}{2H-1} - \frac{1}{2H}\right). & \end{aligned} \tag{A.1}$$

PROOF. We again refer to the extended version of the paper, online at http://arxiv.org/abs/0709.3896v2, for this proof. □

LEMMA 6. *With $f_{1,H}$ given in (3.33) and U_2 in (3.32), we have*

$$\lim_{N\to\infty} \mathbf{E}[(\sqrt{N} U_2)^2] = f_{1,H}.$$

PROOF. We have seen that $\sqrt{c_{3,H}} = 4d(H)$. We have also defined

$$\sqrt{N} U_2 = N^{H-1/2} \sqrt{c_{3,H}} \left[\frac{N^{1-H}}{\sqrt{c_{3,H}}} T_2 - Z(1) \right].$$

Let us simply compute the L^2-norm of the term in brackets. Since this expression is a member of the second chaos and, more specifically, since $T_2 = I_2(f_N)$ and $Z(1) = I_2(L_1)$, where f_N [given in (3.30)] and L_1 [given in (3.3)] are symmetric functions in $L^2([0,1]^2)$, it holds that

$$\begin{aligned} &\mathbf{E}\left[\left(\frac{N^{1-H}}{\sqrt{c_{3,H}}} T_2 - Z(1) \right)^2 \right] \\ &\quad = \left\| \frac{N^{1-H}}{4d(H)} f_N - L_1 \right\|^2_{L^2([0,1]^2)} \\ &\quad = \frac{N^{2-2H}}{4d(H)^2} \|f_N\|_{L^2([0,1]^2)} \\ &\qquad - 2 \frac{N^{1-H}}{4d(H)} \langle f_N, L_1 \rangle_{L^2([0,1]^2)} + \|L_1\|^2_{L^2([0,1]^2)}. \end{aligned}$$

The first term has already been computed. It gives

$$\begin{aligned} &\frac{N^{2-2H}}{4d(H)^2} \|f_N\|_{L^2([0,1]^2)} \\ &\quad = N^{-2H} a^4(H) d^2(H) \\ &\qquad \times \sum_{i,j=1}^{N} \int_{[0,1]^4} du\, dv\, du'\, dv' \\ &\qquad\qquad \times (|u-v||u'-v'||u-u'+i-j||v-v'+i-j|)^{2H'-2}. \end{aligned}$$

By using the expression for the kernel L_1 and Fubini's theorem, the scalar product of f_N and L_1 gives

$$\begin{aligned} &\frac{N^{1-H}}{4d(H)} \langle f_N, L_1 \rangle_{L^2([0,1]^2)} \\ &\quad = \int_0^1 \int_0^1 dy_1\, dy_2\, \frac{N^{1-H}}{4d(H)} f_N(y_1, y_2) L_1(y_1, y_2) \\ &\quad = N^H a(H)^3 d(H)^2 \sum_{i=1}^{N} \int_{I_i} \int_{I_i} du\, dv \int_0^1 du' (|u-v||u-u'||v-u'|)^{2H'-2} \\ &\quad = N^H a(H)^3 d(H)^2 \sum_{i,j=1}^{N} \int_{I_i} \int_{I_i} du\, dv \int_{I_j} du' (|u-v||u-u'||v-u'|)^{2H'-2} \end{aligned}$$

$$= N^{-2H} a(H)^3 d(H)^2 \times \sum_{i,j=1}^{N} \int_{[0,1]^3} (|u-v||u-u'+i-j||v-u'+i-j|)^{2H'-2}\, du\, dv\, du'.$$

Finally, the last term $\|L_1\|^2_{L^2([0,1]^2)}$ can be written in the following way:

$$\begin{aligned}\|L_1\|^2_{L^2([0,1]^2)} &= d(H)^2 a(H)^2 \int_{[0,1]^2} |u-u'|^{2(2H'-2)}\, du\, du' \\ &= d(H)^2 a(H)^2 \sum_{i,j=1}^{N} \int_{I_i} \int_{I_j} |u-u'|^{2(2H'-2)}\, du\, du' \\ &= d(H)^2 a(H)^2 N^{-2H} \sum_{i,j=1}^{N} \int_{[0,1]^2} |u-u'+i-j|^{2(2H'-2)}\, du\, du'.\end{aligned}$$

One can check that, when bringing these three contributions together, the "diagonal" terms corresponding to $i=j$ vanish. Thus, we get

$$\mathbf{E}[(\sqrt{N} U_2)^2] = 32 d(H)^4 a(H)^2 \frac{1}{N} \sum_{k=1}^{N-1} (N-k-1) k^{2H-2} F\left(\frac{1}{k}\right),$$

where F is the function we introduced in (3.34).

This function F is of class C^1 on the interval $[0,1]$. It can be seen that

$$\begin{aligned}F(0) &= \int_{[0,1]^4} du\, dv\, du'\, dv' \\ &\quad \times (a(H)^2 (|u-v||u'-v'|)^{2H'-2} - 2a(H)|u-v| + 1) \\ &= a(H)^2 \left(\int_{[0,1]^2} |u-v|^{2H'-2} \right)^2 - 2a(H) \int_{[0,1]^2} |u-v|^{2H'-2}\, du\, dv + 1 \\ &= 0.\end{aligned}$$

Similarly, one can also calculate the derivative F' and check that $F'(0) = 0$. Therefore, $F(x) = o(x)$ as $x \to 0$. To investigate the sequence $a_N := N^{-1} \sum_{k=1}^{N-1} (N-k-1) k^{2H-2} F(\frac{1}{k})$, we split it into two pieces:

$$\begin{aligned}a_N &= N^{-1} \sum_{k=1}^{N-1} (N-k-1) k^{2H-2} F\left(\frac{1}{k}\right) \\ &= \sum_{k=1}^{N-1} k^{2H-2} F\left(\frac{1}{k}\right) + N^{-1} \sum_{k=1}^{N-1} (k+1) k^{2H-2} F\left(\frac{1}{k}\right) \\ &=: b_N + c_N.\end{aligned}$$

Since b_N is the partial sum of a sequence of positive terms, one only needs to check that the series is finite. The relation $F(1/k) \ll 1/k$ yields that it is finite if and only if $2H-3<-1$, which is true. For the term c_N, one notes that we may replace the factor $k+1$ by k since, by the calculation undertaken for b_N, $N^{-1}\sum_{k=1}^{N-1} k^{2H-2}F(\frac{1}{k})$ converges to 0. Hence, asymptotically, we have

$$c_N \simeq N^{-1}\sum_{k=1}^{N-1} k^{2H-3}F\left(\frac{1}{k}\right) \le N^{-1}\|F\|_\infty \sum_{k=1}^{\infty} k^{2H-3},$$

which thus converges to 0. We have proven that $\lim a_N = \lim b_N = \sum_{k=1}^{\infty} k^{2H-2} \times F(\frac{1}{k})$, which completes the proof of the lemma. $\square$

LEMMA 7. *Defining*

$$g_N(y_1,y_2) := \frac{N^{H-1/2}}{\sqrt{f_{1,H}}}\left(\frac{N^{1-H}}{4d(H)} f_N(y_1,y_2) - L_1(y_1,y_2)\right),$$

we have $\lim_{N\to\infty}\|g_N \otimes_1 g_N\|^2_{L^2([0,1]^2)} = 0$ *provided* $H<2/3$.

PROOF. We omit the leading constant $f_{1,H}^{-1/2}$, which is irrelevant. Using the expression (3.30) for f_N, we have

$$\begin{aligned} g_N(y_1,y_2) &= N^{2H-1/2} d(H)a(H) \\ &\quad \times \sum_{i=1}^{N}\int_{I_i}\int_{I_i} \partial_1 K^{H'}(u,y_1)\,\partial_1 K^{H'}(v,y_2)|u-v|^{2H'-2}\,dv\,du \\ &\quad - L_1(y_1,y_2). \end{aligned}$$

Here, and below, we will be omitting indicator functions of the type $1_{[0,(i+1)/N]}(y_1)$ because, as stated earlier, these are implicitly contained in the support of $\partial_1 K^{H'}$. By decomposing the expression for L_1 from (3.3) over the same blocks $I_i \times I_i$ as for f_N, we can now express the contraction $g_N \otimes_1 g_N$ as follows:

$$(g_N \otimes_1 g_N)(y_1,y_2) = N^{2H-1}(A_N - 2B_N + C_N),$$

where we have introduced three new quantities,

$$\begin{aligned} A_N &:= N^{2H} d(H)^2 a(H)^3 \\ &\quad \times \sum_{i,j=1}^{N}\int_{I_i}\int_{I_i} dv\,du \int_{I_j}\int_{I_j} dv'\,du' \\ &\quad \times [|u-v|\cdot|u'-v'|\cdot|v-v'|]^{2H'-2} \end{aligned}$$

$$\times \partial_1 K^{H'}(u,y_1)\partial_1 K^{H'}(u',y_2),$$

$$B_N := N^H a(H)^2 d(H)^2 \sum_{i=1}^{N} \int_{I_i}\int_{I_i} dv\,du \int_0^1 du'[|u-v|\cdot|u'-v|]^{2H'-2}$$
$$\times \partial_1 K^{H'}(u,y_1)\,\partial_1 K^{H'}(u',y_2)$$
$$= N^H a(H)^2 d(H)^2 \sum_{i,j=1}^{N} \int_{I_i}\int_{I_i} dv\,du \int_{I_j} du'[|u-v|\cdot|u'-v|]^{2H'-2}$$
$$\times \partial_1 K^{H'}(u,y_1)\,\partial_1 K^{H'}(u',y_2)$$

and

$$C_N = d(H)^2 a(H) \int_0^1\int_0^1 dv\,du\,\partial_1 K^{H'}(u,y_1)\,\partial_1 K^{H'}(v,y_2)|u-v|^{2H'-2}$$
$$= d(H)^2 a(H) \sum_{i,j=1}^{N} \int_{I_i}\int_{I_j} dv\,du\,\partial_1 K^{H'}(u,y_1)\,\partial_1 K^{H'}(v,y_2)|u-v|^{2H'-2}.$$

The squared norm of the contraction can then be written as

$$\|g_N \otimes_1 g_N\|^2_{L^2([0,1]^2)}$$
$$= N^{4H-2}(\|A_N\|^2_{L^2([0,1]^2)} + 4\|B_N\|^2_{L^2([0,1]^2)}$$
$$+ \|C_N\|^2_{L^2([0,1]^2)} - 4\langle A_N, B_N\rangle_{L^2([0,1]^2)}$$
$$+ 2\langle A_N, C_N\rangle_{L^2([0,1]^2)} - 4\langle B_N, C_N\rangle_{L^2([0,1]^2)}).$$

Using the definitions of A_N, B_N and C_N, we may express all six terms above explicitly. All of the computations are based on the key relation (3.7).

We obtain

$$\|A_N\|^2_{L^2([0,1]^2)}$$
$$= N^{4H} a(H)^6 d(H)^4 a(H)^2$$
$$\times \sum_{i,j,k,l=1}^{N} \int_{I_i}\int_{I_i} dv\,du \int_{I_j}\int_{I_j} dv'\,du'$$
$$\times \int_{I_k}\int_{I_k} d\bar{u}\,d\bar{v} \int_{I_l}\int_{I_l} d\bar{u}'\,d\bar{v}'$$
$$\times [|u-v|\cdot|u'-v'|\cdot|v-v'|\cdot|\bar{u}-\bar{v}|\cdot|\bar{u}'-\bar{v}'|$$
$$\times |\bar{v}-\bar{v}'|\cdot|u-\bar{u}|\cdot|u'-\bar{u}'|]^{2H'-2}$$
$$= N^{4H} a(H)^8 d(H)^4 \frac{1}{N^8}\frac{1}{N^{8(2H'-2)}}$$

$$\times \sum_{i,j,k,l=1}^{N} \int_{[0,1]^8} du\, dv\, du'\, dv'\, d\bar{u}\, d\bar{v}\, d\bar{u}'\, d\bar{v}'$$

$$\times \, ||u-v|\cdot|u'-v'||\bar{u}-\bar{v}||\bar{u'}-\bar{v}'||^{2H'-2}$$

$$\times [|v-v'+i-j|\cdot|\bar{v}-\bar{v}'+k-l|$$

$$\times |u-\bar{u}+i-k|\cdot|u'-\bar{u}'+j-l|]^{2H'-2},$$

$$\|B_N\|^2_{L^2([0,1]^2)}$$

$$= N^{2H} a(H)^6 d(H)^4$$

$$\times \sum_{i,j,k,l=1}^{N} \int_{I_i}\int_{I_i} dv\, du \int_{I_j} du' \int_{I_k}\int_{I_k} d\bar{u}\, d\bar{v} \int_{I_l} d\bar{u}'$$

$$\times [|u-v|\cdot|u'-v||\bar{u}-\bar{v}|\cdot|\bar{u}'-\bar{v}|\cdot|u-\bar{u}|\cdot|u'-\bar{u}'|]^{2H'-2}$$

$$= N^{2H} a(H)^6 d(H)^4$$

$$\times \sum_{i,j,k,l=1}^{N} \int_{[0,1]^6} du\, dv\, du'\, d\bar{u}\, d\bar{v}\, d\bar{u}'$$

$$\times [|u-v|\cdot|u'-v+i-j||\bar{u}-\bar{v}|\cdot|\bar{u}'-\bar{v}+k-l|$$

$$\times |u-\bar{u}+i-k|\cdot|u'-\bar{u}'+j-l|]^{2H'-2}$$

and

$$\|C_N\|^2_{L^2([0,1]^2)}$$

$$= N^{2H} a(H)^4 d(H)^4 \sum_{i,j,k,l=1}^{N} \int_{I_i}\int_{I_j} dv\, du \int_{I_k}\int_{I_l} dv'\, du'$$

$$\times [|u-v|\cdot|u'-v'|\cdot|u-u'|\cdot|v-v'|]^{2H'-2}$$

$$= N^{2H} a(H)^4 d(H)^4 \frac{1}{N^4}\frac{1}{N^{4(2H'-2)}}$$

$$\times \sum_{i,j,k,l=1}^{N} \int_{[0,1]^4} du\, dv\, du'\, dv'$$

$$\times [|u-v+i-j|\cdot|u'-v'+k-l|$$

$$\times |u-u'+i-k|\cdot|v-v'+j-l|]^{2H'-2}.$$

The inner product terms can be also treated in the same manner. First,

$\langle A_N, B_N\rangle_{L^2([0,1]^2)}$

$$
\begin{aligned}
&= N^{3H} a(H)^7 d(H)^4 \\
&\quad \times \sum_{i,j,k,l=1}^{N} \int_{I_i}\int_{I_i} du\, dv \int_{I_j}\int_{I_j} du'\, dv' \int_{I_k}\int_{I_k} d\bar{u}\, d\bar{v} \int_{I_l} d\bar{u}' \\
&\quad \times [|u-v| \cdot |u'-v'| \cdot |v-v'| \cdot |\bar{u}-\bar{v}| \cdot |\bar{u}'-\bar{v}| \cdot |u-\bar{u}| \cdot |u'-\bar{u}'|]^{2H'-2} \\
&= N^{3H} a(H)^7 d(H)^4 \frac{1}{N^7} \frac{1}{N^{7(2H'-2)}} \\
&\quad \times \sum_{i,j,k,l=1}^{N} \int_{[0,1]^7} du\, dv\, du'\, dv'\, d\bar{u}\, d\bar{v}\, d\bar{u}' \\
&\qquad\qquad \times [|u-v| \cdot |u'-v'| \cdot |v-v'+i-j| \cdot |\bar{u}-\bar{v}| \\
&\qquad\qquad\quad \times |\bar{u}'-\bar{v}+k-l| \cdot |u-\bar{u}+i-k| \cdot |u'-\bar{u}'+j-l|]^{2H'-2}
\end{aligned}
$$

and

$$
\begin{aligned}
&\langle A_N, C_N \rangle_{L^2([0,1]^2)} \\
&\quad = N^{2H} a(H)^6 d(H)^4 \\
&\qquad \times \sum_{i,j,k,l=1}^{N} \int_{I_i}\int_{I_i} du\, dv \int_{I_j}\int_{I_j} du'\, dv' \int_{I_k} d\bar{u} \int_{I_l} d\bar{v} \\
&\qquad \times [|u-v| \cdot |u'-v'| \cdot |v-v'| \cdot |\bar{u}-\bar{v}| \cdot |u-\bar{u}| \cdot |u'-\bar{v}]^{2H'-2} \\
&\quad = N^{2H} a(H)^6 d(H)^4 \frac{1}{N^6} \frac{1}{N^{6(2H'-2)}} \\
&\qquad \times \sum_{i,j,k,l=1}^{N} \int_{[0,1]^6} du\, dv\, du'\, dv'\, d\bar{u}\, d\bar{v} \\
&\qquad\qquad \times [|u-v| \cdot |u'-v'| \cdot |v-v'+i-j| \\
&\qquad\qquad\quad \times |u-\bar{u}+i-k| \cdot |\bar{u}-\bar{v}+k-l| \cdot u'-\bar{v}]^{2H'-2}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&\langle B_N, C_N \rangle_{L^2([0,1]^2)} \\
&\quad = N^{H} a(H)^3 d(H)^4 \\
&\qquad \times \sum_{i,j,k,l=1}^{N} \int_{I_i}\int_{I_i} du\, dv \int_{I_j} du' \int_{I_k} d\bar{u} \int_{I_l} d\bar{v} \\
&\qquad \times [|u-v| \cdot |u'-v| \cdot |\bar{u}-\bar{v}| \cdot |u-\bar{u}| \cdot |u'-\bar{v}|]^{2H'-2}
\end{aligned}
$$

$$
\begin{aligned}
&= N^H a(H)^3 d(H)^4 \frac{1}{N^5} \frac{1}{N^{5(2H'-2)}} \\
&\quad \times \sum_{i,j,k,l=1}^{N} \int_{[0,1]^5} du\, dv\, du'\, d\bar{u}\, d\bar{v} \\
&\qquad \times [|u-v| \cdot |u'-v+i-j| \cdot |\bar{u}-\bar{v}+k-l| \\
&\qquad\qquad \times |u-\bar{u}+i-k| \cdot |u'-\bar{v}+j-l|]^{2H'-2}.
\end{aligned}
$$

We now summarize our computations. Note that the factors $d(H)^4$ and $\frac{1}{N^4}\frac{1}{N^{4(2H'-2)}}$ are common to all terms. We also note that any terms corresponding to difference of indices smaller than 3 can be shown to tend collectively to 0, similarly for other "diagonal" terms in this study. The proof is omitted. We thus assume that the sums over the set D of indices i, j, k, l in $\{1, \ldots, N\}$ such that $|i-j|$, $|k-l|$, $|i-k|$ and $|j-l|$ are all at least 2. Hence, we get

$$
\begin{aligned}
&\|g_N \otimes_1 g_N\|^2_{L^2([0,1]^2)} \\
&\quad = d(H)^4 N^{4H-2} \frac{1}{N^4} \\
&\qquad \times \sum_{(i,j,k,l)\in D} \left(\frac{|i-j| \cdot |k-l| \cdot |i-k| \cdot |j-l|}{N^4} \right)^{2H'-2} \\
&\qquad\qquad \times G\left(\frac{1}{i-j}, \frac{1}{k-l}, \frac{1}{i-k}, \frac{1}{j-l} \right),
\end{aligned}
\tag{A.2}
$$

where the function G is defined for $(x, y, z, w) \in [1/2, 1/2]^4$ by

$$
\begin{aligned}
&G(x, y, z, w) \\
&\quad = a(H)^8 \int_{[0,1]^8} du\, dv\, du'\, dv'\, d\bar{u}\, d\bar{v}\, d\bar{u}'\, d\bar{v}' \\
&\qquad \times [|u-v| \cdot |u'-v'| \cdot |\bar{u}-\bar{v}| \cdot |\bar{u}'-\bar{v}'|]^{2H'-2} \\
&\qquad \times [|(v-v')x+1| \cdot |(\bar{v}-\bar{v}')y+1| \\
&\qquad\quad \times |(u-\bar{u})z+1| \cdot |(u'-\bar{u}')w+1|]^{2H'-2} \\
&\qquad + 4a(H)^6 \int_{[0,1]^6} du\, dv\, du'\, d\bar{u}\, d\bar{v}\, d\bar{u}' \\
&\qquad\quad \times [|u-v| \cdot |\bar{u}-\bar{v}| \cdot |(u'-v)x+1| \cdot |(\bar{u}'-\bar{v})y+1| \\
&\qquad\qquad \times |(u-u')z+1| \cdot |(u'-\bar{u}')w+1|]^{2H'-2} \\
&\qquad + a(H)^4 \int_{[0,1]^4} du\, dv\, du' dv\, dv'
\end{aligned}
$$

$$
\begin{aligned}
&\times [|(u-v)x+1| \cdot |(u'-v')y+1| \\
&\qquad \times |(u-u')z+1| \cdot |(v-v')w+1|]^{2H'-2} \\
&- 4a(H)^7 \int_{[0,1]^7} du\, dv\, du'\, dv'\, d\bar{u}\, d\bar{v}\, d\bar{u}' \\
&\times [|u-v| \cdot |u'-v'| \cdot |\bar{u}-\bar{v}| \cdot |(v-v')x+1| \\
&\qquad \times |(\bar{u}'-\bar{v})y+1| \cdot |(u-u')z+1| \cdot |(u'-\bar{u}')w+1|]^{2H'-2} \\
&+ 2a(H)^6 \int_{[0,1]^6} du\, dv\, du'\, dv'\, d\bar{u}\, d\bar{v} \\
&\times [|u-v| \cdot |u'-v'| \cdot |(v-v')x+1| \cdot |(\bar{u}-\bar{v})y+1| \\
&\qquad\qquad\qquad \times |(u-u')z+1| \cdot |(u'-\bar{v})w+1|]^{2H'-2} \\
&- 4a(H)^5 \int_{[0,1]^5} du\, dv\, du'\, d\bar{u}\, d\bar{v} \\
&\times [|u-v| \cdot |(v-u')x+1| \cdot |(\bar{u}-\bar{v})y+1| \\
&\qquad\qquad \times |(u-\bar{u})z+1| \cdot |(u'-\bar{v})w+1|]^{2H'-2}.
\end{aligned}
$$

It is elementary to check that G and all its partial derivatives are bounded on $[-1/2,1/2]^4$. More specifically, by using the identity

$$a(H)^{-1} = \int_0^1 \int_0^1 |u-v|^{2H'-2}\, du\, dv,$$

we obtain

$$
\begin{aligned}
G(0,0,0,0) &= a(H)^4 + 4a(H)^4 + a(H)^4 - 4a(H)^4 + 2a(H)^4 - 4a(H)^4 \\
&= 0.
\end{aligned}
$$

The boundedness of G's partial derivatives implies, by the mean value theorem, that there exists a constant K such that, for all $(i,j,k,l) \in D$,

$$
\begin{aligned}
&\left| G\left(\frac{1}{i-j}, \frac{1}{k-l}, \frac{1}{i-k}, \frac{1}{j-l}\right) \right| \\
&\qquad \le \frac{K}{|i-j|} + \frac{K}{|k-l|} + \frac{K}{|i-k|} + \frac{K}{|j-l|}.
\end{aligned}
$$

Hence, from (A.2), because of the symmetry of the sum with respect to the indices, it is sufficient to show that the following converges to 0:

$$S := N^{4H-2} \frac{1}{N^4} \sum_{(i,j,k,l) \in D} \left(\frac{|i-j| \cdot |k-l| \cdot |i-k| \cdot |j-l|}{N^4} \right)^{H-1} \frac{1}{|i-j|}. \tag{A.3}$$

We will express this quantity by singling out the term $i' := i - j$ and summing over it last:

$$S = 2N^{4H-1} \sum_{i'=3}^{N-1} \frac{1}{N^3} \sum_{\substack{(i'+j,j,k,l)\in D \\ 1\le j\le N-i'}} \left(\frac{|k-l|\cdot|i'+j-k|\cdot|j-l|}{N^3}\right)^{H-1} \left(\frac{i'}{N}\right)^{H-1} \frac{1}{i'}$$

$$= 2N^{3H-2} \sum_{i'=3}^{N-1} (i')^{H-2} \frac{1}{N^3} \sum_{\substack{(i'+j,j,k,l)\in D \\ 1\le j\le N-i'}} \left(\frac{|k-l|\cdot|i'+j-k|\cdot|j-l|}{N^3}\right)^{H-1}.$$

For fixed i', we can compare the sum over j,k,l to a Riemann integral since the power $H-1>-1$. This cannot be done, however, for $(i')^{H-2}$; rather, one must use the fact that this is the term of a summable series. We get that, asymptotically for large N,

$$S \simeq 2N^{3H-2} \sum_{i'=3}^{N-1} (i')^{H-2} g(i'/N),$$

where the function g is defined on $[0,1]$ by

$$g(x) := \int_0^{1-x} \int_0^1 \int_0^1 dy\, dz\, dw\, |(z-w)(x+y-z)(y-w)|^{H-1}. \tag{A.4}$$

It is easy to check that g is a bounded function on $[0,1]$; thus, we have proven that for some constant $K>0$,

$$S \le KN^{3H-2} \sum_{i'=3}^{\infty} (i')^{H-2},$$

which converges to 0 provided $H<2/3$. This completes the proof of the lemma. $\square$

We conclude this appendix with a discussion of why the threshold $H<2/3$ cannot be improved upon, and the consequences of this. We can perform a finer analysis of the function G in the proof above. The first and second derivatives of G at $\bar{0} = (0,0,0,0)$ can be calculated by hand. The calculation is identical for $\partial G/\partial x(\bar{0})$ and all other first derivatives, yielding [via the expression used above for $a(H)$],

$$\begin{aligned}
&\frac{1}{H-1}\frac{\partial G}{\partial x}(\bar{0}) \\
&\quad = a(H)^6 \int_{[0,1]^4} du\, dv\, du'\, dv' (v-v')[|u-v|\cdot|u'-v'|]^{H-1} \\
&\quad\quad + 4a(H)^5 \int_{[0,1]^3} du\, dv\, du' (v-u')|u-v|^{H-1}
\end{aligned}$$

$$+ a(H)^4 \int_{[0,1]^2} du\, dv (u-v)$$
$$- 4a(H)^6 \int_{[0,1]^4} du\, dv\, du'\, dv' (v-v')[|u-v| \cdot |u'-v'|]^{H-1}$$
$$+ 2a(H)^6 \int_{[0,1]^4} du\, dv\, du'\, dv' (v-v')[|u-v| \cdot |u'-v'|]^{H-1}$$
$$- 4a(H)^5 \int_{[0,1]^3} du\, dv\, du' (v-u')|u-v|^{H-1}.$$

We note that the two lines with $4a(H)^5$ cancel each other out. For each of the other four lines, we see that the factor $(v-v')$ is an odd term and the other factor is symmetric with respect to v and v'. Therefore, each of the other four factors is zero individually. This proves that the gradient of G at 0 is null. Let us find expressions for the second derivatives. Similarly to the above calculation, we can write

$$\frac{1}{(1-H)(2-H)} \frac{\partial^2 G}{\partial x^2}(\bar{0})$$
$$= a(H)^6 \int_{[0,1]^4} du\, dv\, du'\, dv' (v-v')^2[|u-v| \cdot |u'-v'|]^{H-1}$$
$$+ 4a(H)^5 \int_{[0,1]^3} du\, dv\, du' (v-u')^2|u-v|^{H-1}$$
$$+ a(H)^4 \int_{[0,1]^2} du\, dv (u-v)^2$$
$$- 4a(H)^6 \int_{[0,1]^4} du\, dv\, du'\, dv' (v-v')^2[|u-v| \cdot |u'-v'|]^{H-1}$$
$$+ 2a(H)^6 \int_{[0,1]^4} du\, dv\, du'\, dv' (v-v')^2[|u-v| \cdot |u'-v'|]^{H-1}$$
$$- 4a(H)^5 \int_{[0,1]^3} du\, dv\, du' (v-u')^2|u-v|^{H-1}.$$

Again, the terms with $a(H)^5$ cancel each other out. The three terms with $a(H)^6$ add to a nonzero value and we thus get

$$\frac{1}{(1-H)(2-H)} \frac{\partial^2 G}{\partial x^2}(\bar{0})$$
$$= -a(H)^6 \int_{[0,1]^4} du\, dv\, du'\, dv' (v-v')^2[|u-v| \cdot |u'-v'|]^{H-1}$$
$$+ a(H)^4 \int_{[0,1]^4} du\, dv (u-v)^2.$$

While the evaluation of this integral is nontrivial, we can show that for all $H > 1/2$, it is a strictly positive constant $\gamma(H)$. Similar computations can be attempted for the mixed derivatives, which are all equal to some common value $\eta(H)$ at $\bar{0}$ because of G's symmetry, and we will see that the sign of $\eta(H)$ is irrelevant. We can now write, using Taylor's formula,

$$\begin{aligned} G(x,y,z,w) = {} & \gamma(H)(x^2+y^2+z^2+w^2) \\ & + \eta(H)(xy+xz+xw+yz+yw+zw) \\ & + o(x^2+y^2+z^2+w^2). \end{aligned}$$

By taking $x^2+y^2+z^2+w^2$ sufficiently small [this corresponds to restricting $|i-j|$ and other differences to being larger than some value $m = m(H)$, whose corresponding "diagonal" terms not satisfying this restriction are dealt with as usual], we get, for some constant $\theta(H) > 0$,

$$G(x,y,z,w) \geq \theta(H)(x^2+y^2+z^2+w^2) + \eta(H)(xy+xz+xw+yz+yw+zw).$$

Let us first look at the terms in (A.2) corresponding to $x^2+y^2+z^2+w^2$. These are collectively bounded below by the same sum restricted to $i = j+m$, which equals

$$d(H)^4 N^{4H-2} \frac{1}{N^4} \sum_{(j+m,j,k,l)\in D} \left(\frac{|i-j|\cdot|k-l|\cdot|i-k|\cdot|j-l|}{N^4} \right)^{2H'-2} \frac{\theta(H)}{(i-j)^2}.$$

The fact that the final factor contains $(i-j)^{-2}$ instead of $(i-j)^{-1}$, which we had, for instance, in (A.3) in the proof of the lemma, does not help us. In particular, calculations identical to those following (A.3) show that the above is larger than

$$2N^{3H-2} g(m/N),$$

which does not go to 0 if $H \geq 2/3$ since $g(0)$ calculated from (A.4) is positive.

For the terms in (A.2) corresponding to $xy+xz+xw+yz+yw+zw$, considering, for instance, the term xy, similar computations to those above lead to the corresponding term in S being equal to

$$\begin{aligned} & 2N^{2H-2} \sum_{i'=m}^{N-1} \sum_{k'=m}^{N-1} (i'k')^{H-2} \frac{1}{N^2} \\ & \qquad \sum_{\substack{(i'+j,j,k'+l,l)\in D \\ 1\leq j\leq N-i';1\leq l\leq N-k'}} \left(\frac{|i'+j-k'-l|\cdot|j-l|}{N^3} \right)^{H-1} \\ & \quad \simeq 2N^{2H-2} \sum_{i'=m}^{N-1} \sum_{k'=m}^{N-1} (i'k')^{H-2} \int_0^{1-i'/N} \int_0^{1-k'/N} dy\,dw \\ & \qquad\qquad \times \left| (z-w)\left(\frac{i'}{N} + y - \frac{k'}{N} - w \right)(y-w) \right|^{H-1}, \end{aligned}$$

which evidently tends to 0 provided $H < 1$.

We conclude that if $H \geq 2/3$, then $\|g_N \otimes_1 g_N\|^2_{L^2([0,1]^2)}$ does not tend to 0 and, by the Nualart–Ortiz-Latorre criterion [Theorem 2.1 part (iii)], U_2, as defined in (3.32), does not converge in distribution to a normal. Hence, we can guarantee that, provided $H \geq 2/3$, the adjusted variation in Theorem 3.3 does not converge to a normal. Thus, the normality of our adjusted estimator in Theorem 4.1 holds *if and only if* $H \in (1/2, 2/3)$.

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